

# Steady hydrodynamic model of semiconductors with sonic boundary

Jingyu Li<sup>1</sup>, Ming Mei<sup>2,3</sup>, Guojing Zhang<sup>1\*</sup> and Kaijun Zhang<sup>1</sup>

<sup>1</sup>*School of Mathematics and Statistics, Northeast Normal University,  
Changchun 130024, P.R.China*

<sup>2</sup>*Department of Mathematics, Champlain College Saint-Lambert,  
Saint-Lambert, Quebec, J4P 3P2, Canada*

<sup>3</sup>*Department of Mathematics and Statistics, McGill University,  
Montreal, Quebec, H3A 2K6, Canada*

## Abstract

In this paper, we study the well-posedness/ill-posedness and regularity of stationary solutions to the hydrodynamic model of semiconductors represented by Euler-Poisson equations with sonic boundary, and make a classification on these solutions. When the doping profile is subsonic, we prove that, the corresponding steady-state equations with sonic boundary possess a unique interior subsonic solution, and at least one interior supersonic solution; and if the relaxation time is large and the doping profile is a small perturbation of constant, then the equations admit infinitely many interior transonic shock solutions; while, if the relaxation time is small enough and the doping profile is a subsonic constant, then the equations admits infinitely many interior  $C^1$  smooth transonic solutions, and no transonic shock solution exists. When the doping profile is supersonic, we show that the system does not hold any subsonic solution; furthermore, the system doesn't admit any supersonic solution or any transonic solution if such a supersonic doping profile is small enough or the relaxation time is small, but it has at least one supersonic solution and infinitely many transonic solutions if the supersonic doping profile is close to the sonic line and the relaxation time is large. The interior subsonic/supersonic solutions all are globally  $C^{\frac{1}{2}}$  Hölder-continuous, and the Hölder exponent  $\frac{1}{2}$  is optimal. The non-existence of any type solutions in the case of small doping profile or small relaxation time indicates that the semiconductor effect for the system is remarkable and cannot be ignored. The proof for the existence of subsonic/supersonic solutions is the technical compactness analysis combining the energy method and the phase-plane analysis, while the approach for the existence of multiple transonic solutions is artfully constructed. The results obtained significantly improve and develop the existing studies.

**Keywords:** Euler-Poisson equations, hydrodynamic model of semiconductors, sonic boundary, subsonic solutions, supersonic solutions, transonic solutions with shock,  $C^1$  smooth transonic solution.

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\*Corresponding author.

E-mail addresses: lijy645@nenu.edu.cn (J. Li), ming.mei@mcgill.ca (M. Mei), zhanggj100@nenu.edu.cn (G. Zhang), zhangkj201@nenu.edu.cn (K. Zhang)

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## 1 Introduction

The hydrodynamic model of semiconductors, first introduced by Bløtekjær in [5], is usually described for the charged fluid particles such as electrons and holes in semiconductor devices [5, 19, 23] and positively and negatively charged ions in plasma [27]. The governing equations are Euler-Poisson equations as follows [15, 16, 17, 20]:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + P(\rho))_x = \rho E - \frac{\rho u}{\tau}, \\ E_x = \rho - b(x). \end{cases} \quad (1.1)$$

Here  $\rho$ ,  $u$  and  $E$  represent the electron density, the velocity and the electric field, respectively.  $P(\rho)$  is the pressure function of the electron density. When the system is isothermal, the pressure function is physically represented by

$$P(\rho) = T\rho, \quad \text{with the constant temperature } T > 0. \quad (1.2)$$

The function  $b(x) > 0$  is the doping profile standing for the density of impurities in semiconductor device. The constant  $\tau > 0$  denotes the momentum relaxation time.

In this series of study, we are mainly interested in investigating the existence of the solutions to (1.1) with sonic boundary, and the large-time behavior of the solutions. At the first but important stage, we focus on the existence and classification of all stationary solutions to the steady-state system of equations with sonic boundary. This will be the main purpose of the present paper.

In this paper, we consider the following steady-state equations to (1.1) in the bounded domain  $[0, 1]$ . Denote  $J = \rho u$ , the current density, then we have the stationary equations of (1.1) as follows

$$\begin{cases} J = \text{constant}, \\ \left( \frac{J^2}{\rho} + P(\rho) \right)_x = \rho E - \frac{J}{\tau}, \\ E_x = \rho - b(x). \end{cases} \quad x \in (0, 1). \quad (1.3)$$

Using the terminology from gas dynamics, we call  $c := \sqrt{P'(\rho)} = \sqrt{T} > 0$  the sound speed for  $P(\rho) = T\rho$  (see (1.2)). Thus, the stationary flow of (1.3) is called to be subsonic/sonic/supersonic, if the fluid velocity satisfies

$$\text{fluid velocity: } u = \frac{J}{\rho} \begin{matrix} \leq \\ \equiv \\ \geq \end{matrix} c = \sqrt{P'(\rho)} = \sqrt{T} : \text{ sound speed.} \quad (1.4)$$

We consider the current driven flow, thus the current density  $J$  is a prescribed constant. Note that if  $(\rho(x), E(x))$  is a solution to (1.3) with a given constant current density  $J$ , then  $(\rho(1-x), -E(1-x))$  is a solution to (1.3) with respect to  $-J$  and  $b(1-x)$ . So, we may consider only the case of  $J > 0$ . Without loss of generality, let us assume throughout the paper

$$T = J = 1.$$

Thus, (1.3) is transformed to

$$\begin{cases} \left( 1 - \frac{1}{\rho^2} \right) \rho_x = \rho E - \frac{1}{\tau}, \\ E_x = \rho - b(x). \end{cases} \quad (1.5)$$

From (1.4), it can be identified that,  $\rho > 1$  is for the subsonic flow,  $\rho = 1$  stands for the sonic flow, and  $0 < \rho < 1$  represents for the supersonic flow. Therefore, our sonic boundary conditions to (1.3) are proposed as follows

$$\text{sonic boundary: } \rho(0) = \rho(1) = 1. \quad (1.6)$$

Dividing the first equation of (1.5) by  $\rho$  and differentiating the resultant equation with respect to  $x$ , and substituting the second equation of (1.5) to this modified equation, then we have

$$\begin{cases} \left[ \left( \frac{1}{\rho} - \frac{1}{\rho^3} \right) \rho_x \right]_x + \frac{1}{\tau} \left( \frac{1}{\rho} \right)_x - [\rho - b(x)] = 0, & x \in (0, 1), \\ \rho(0) = \rho(1) = 1 \text{ (sonic boundary)}. \end{cases} \quad (1.7)$$

When  $\rho(x) > 1$  or  $0 < \rho(x) < 1$  for  $x \in (0, 1)$ , the equation (1.7) is elliptic but degenerate at the sonic boundary. When  $\rho(x) > 0$  varies around the sonic line  $\rho = 1$  for  $x \in (0, 1)$ , the system then changes its property and occurs phase transitions. The degeneracy of (1.7) on the

boundary will cause us some essential difficulty in the study of well-posedness and regularity of the solutions, and the phenomena of structure of solutions are really rich and interesting.

Throughout the paper we assume that the doping profile  $b(x) \in L^\infty(0, 1)$  and denote

$$\underline{b} := \operatorname{ess\,inf}_{x \in (0,1)} b(x) \quad \text{and} \quad \bar{b} := \operatorname{ess\,sup}_{x \in (0,1)} b(x).$$

Now we introduce the concepts of interior subsonic/supersonic/transonic solutions.

**Definition 1.1.**  $\rho(x)$  is called an interior subsonic (correspondingly, interior supersonic) solution of equation (1.7), if  $\rho(0) = \rho(1) = 1$  but  $\rho(x) \geq 1$  (correspondingly,  $0 < \rho(x) \leq 1$ ) for  $x \in (0, 1)$ , and  $(\rho(x) - 1)^2 \in H_0^1(0, 1)$ , and it holds that for any  $\varphi \in H_0^1(0, 1)$

$$\int_0^1 \left( \frac{1}{\rho} - \frac{1}{\rho^3} \right) \rho_x \varphi_x dx + \frac{1}{\tau} \int_0^1 \frac{\varphi_x}{\rho} dx + \int_0^1 (\rho - b) \varphi dx = 0,$$

which is equivalent to

$$\frac{1}{2} \int_0^1 \frac{\rho + 1}{\rho^3} ((\rho - 1)^2)_x \varphi_x dx + \frac{1}{\tau} \int_0^1 \frac{\varphi_x}{\rho} dx + \int_0^1 (\rho - b) \varphi dx = 0. \quad (1.8)$$

Once  $\rho = \rho(x)$  is determined by equation (1.7), in view of the first equation of (1.5), the electric field  $E(x)$  can be solved by

$$E(x) = \left( \frac{1}{\rho} - \frac{1}{\rho^3} \right) \rho_x + \frac{1}{\tau \rho} = \frac{(\rho + 1)[(\rho - 1)^2]_x}{2\rho^3} + \frac{1}{\tau \rho}.$$

In this way, we could obtain the interior subsonic/supersonic solutions to system (1.5)-(1.6).

**Definition 1.2.**  $\rho(x) > 0$  is called a  $C^1$  transonic solution of system (1.5)-(1.6), if  $\rho(x) \in C^1(0, 1)$  with  $\rho(0) = \rho(1) = 1$  and there exists a number  $x_0 \in (0, 1)$  such that

$$\rho(x) = \begin{cases} \rho_{sup}(x), & x \in (0, x_0), \\ \rho_{sub}(x), & x \in (x_0, 1), \end{cases}$$

where  $0 < \rho_{sup}(x) \leq 1$ ,  $\rho_{sub}(x) \geq 1$  and

$$\rho_{sup}(x_0) = \rho_{sub}(x_0) \quad \text{and} \quad \rho'_{sup}(x_0) = \rho'_{sub}(x_0). \quad (1.9)$$

$\rho(x) > 0$  is called a transonic shock solution of system (1.5)-(1.6), if  $\rho(0) = \rho(1) = 1$  and it is separated at a point  $x_0 \in (0, 1)$  in the form

$$\rho(x) = \begin{cases} \rho_{sup}(x), & x \in (0, x_0), \\ \rho_{sub}(x), & x \in (x_0, 1), \end{cases}$$

where  $0 < \rho_{sup}(x) \leq 1$  and  $\rho_{sub}(x) \geq 1$  satisfies the entropy condition at  $x_0$

$$0 < \rho_{sup}(x_0^-) < 1 < \rho_{sub}(x_0^+), \quad (1.10)$$

and the Rankine-Hugoniot condition

$$\begin{aligned}\rho_{sup}(x_0^-) + \frac{1}{\rho_{sup}(x_0^-)} &= \rho_{sub}(x_0^+) + \frac{1}{\rho_{sub}(x_0^+)}, \\ E_{sup}(x_0^-) &= E_{sub}(x_0^+).\end{aligned}\tag{1.11}$$

Set  $\rho_l = \rho_{sup}(x_0^-)$  and  $\rho_r = \rho_{sub}(x_0^+)$ , a simple computation from (1.11) shows that

$$\rho_l \rho_r = 1.\tag{1.12}$$

The existence of subsonic/supersonic/transonic solutions to the steady-state Euler-Poisson equations for the hydrodynamic model of semiconductors has been intensively studied. In 1990, Degond and Markowich [9] first showed the existence of subsonic solution when the flow and its boundary are completely subsonic. The uniqueness was obtained with a very strong subsonic background, namely,  $|J| \ll 1$ . Then, the steady subsonic flows were deeply studied with different boundaries as well as the higher dimensions case in [2, 3, 10, 11, 15, 18, 24], see the references and therein. For the case of steady supersonic flows, Peng and Violet [25] obtained the existence and uniqueness of supersonic solution when the boundary is with a strongly supersonic background (i.e.  $J \gg 1$ ). On the other hand, the case of steady transonic flows has been also paid a lot of attention. By a phase-plane analysis, Ascher *et al* [1] first tested the existence of transonic solution when the boundary is subsonic but the constant background charge  $b(x)$  is supersonic, which was then extended by Rosini [26] for a bit general case. On the other hand, by using the vanishing viscosity limit method, Gamba constructed 1-D transonic solutions with transonic shocks in [12], and 2-D transonic solutions in [13], but the solutions as the limits of vanishing viscosity yield boundary layers. Recently, Luo-Xin [22] and Luo-Rauch-Xie-Xin [21] studied the hydrodynamic model (1.1) of Euler-Poisson equations without the effect of semiconductor, namely, the momentum equation (1.1)<sub>2</sub> is missing the term of  $-\frac{J}{\tau}$ . This means the current density  $J = 0$  (the absence of semiconductor effect for the device), or the relaxation time  $\tau = \infty$  (the huge relaxation time). Some interesting results on the structure of steady solutions with non-sonic boundary are obtained. Precisely, based on phase-plane analysis, Luo-Xin [22] thoroughly studied the existence/non-existence, uniqueness/non-uniqueness of the transonic solutions with one side supersonic boundary and the other side subsonic boundary when the doping profile  $b(x)$  is a constant either in the supersonic regime or the subsonic regime. Some restrictions on the boundary and the domain are also needed. Then, Luo-Rauch-Xie-Xin [21] showed the existence of the transonic solution to the variable doping profile  $b(x)$  which is regarded as a small perturbation of the constant doping profile  $b(x) \equiv b$ , and further proved the time-asymptotic stability of the transonic shock profiles.

In this paper, the model considered is with the semiconductor effect, and the boundary is, in particular, sonic. These features make the study more difficult and different from the existing studies. In fact, the elliptic equation (1.7) is degenerate at the boundary, but the equations considered in the previous studies are uniformly elliptic whatever in the supersonic regime [25] or the subsonic regime [9]. On the other hand, when the doping profile is sonic or supersonic, we realize that there is no any physical solution if the doping profile is small or the relaxation time is small, which is totally different from the studies [22, 21] in the case without the semiconductor effect. In fact, this demonstrates that the semiconductor effect is remarkable and cannot be ignored.

The main purpose in this paper is to prove the well-posedness/ill-posedness of the steady Euler-Poisson equations (1.5) with the sonic boundary (1.6), and the regularity of subsonic and supersonic solutions when these solutions exist, and the property of the infinitely many transonic solutions. Precisely speaking, when the doping profile is subsonic, we prove that, the corresponding steady-state equations with sonic boundary possess a unique interior subsonic solution, and at least one interior supersonic solution, and infinitely many interior transonic shock solutions for  $\tau \gg 1$  (the small effect of semiconductor), where the case  $\tau = \infty$  studied in [22, 21] is our special case; while, when  $\tau \ll 1$  (the large effect of semiconductor), the system possesses infinitely many  $C^1$  transonic solutions, and no transonic shocks exist. Note that, the transonic shocks have been intensively studied in [1, 12, 13, 22, 21], but, to our best knowledge, the  $C^1$  transonic solutions in semiconductor models are first obtained in the present paper. Essentially, the strong damping effect makes the transonic solutions to be  $C^1$  smooth. Recall that  $C^2$  transonic flow also arises in finite de Laval nozzles, where the geometry structure of the nozzle causes the transonic flow to be  $C^2$  smooth (see the interesting work of C. Wang and Z. Xin [30, 31]). On the other hand, when the doping profile is supersonic, we show that the system does not hold any subsonic solution; and the system also has no supersonic solution and no transonic solution if such a supersonic doping profile is small enough or the relaxation time is large; but it possesses at least one supersonic solution and infinitely many transonic shock solutions if the supersonic doping profile is close to the sonic line and the semiconductor effect is small. When the doping profile is sonic, then the system exists the sonic solution. In all cases mentioned above, all interior subsonic/supersonic solutions obtained are proved to be globally  $C^{\frac{1}{2}}$  Hölder continuous, and the  $C^{\frac{1}{2}}$ -regularity is optimal. We notice that the same regularity  $C^{\frac{1}{2}}$  was also obtained for the subsonic-sonic flow for the steady nozzle in [28, 29]. Regarding the other interesting studies on the subsonic-sonic flow for the steady nozzle, we refer to [4, 8, 7, 32]. To prove the existence of the subsonic/supersonic/transonic solutions and their regularity are non-trivial, because the degeneracy of ellipticity for equation (1.7) at the sonic boundary causes

us essential difficulty. Here, for the existence of subsonic/supersonic solutions to equations (1.5) and (1.6), the proof adopted is the technical compactness analysis combining the energy method with the help of the phase-plane analysis, while for the existence of multiple transonic shock solutions and  $C^1$ -smooth transonic solutions, the approach is the artful construction method. These results are presented in the following Theorems 1.1-1.2, which essentially improve and develop the existing studies.

**Theorem 1.1** (Case of subsonic doping profile). *Let the doping profile be subsonic such that  $b(x) \in L^\infty(0, 1)$  and  $\underline{b} > 1$ . Then the steady-state Euler-Poisson equations (1.5) and (1.6) admit:*

1. *A unique pair of interior subsonic solution  $(\rho_{sub}, E_{sub})(x) \in C^{\frac{1}{2}}[0, 1] \times H^1(0, 1)$  satisfying*

$$1 + m \sin(\pi x) \leq \rho_{sub}(x) \leq \bar{b}, \quad x \in [0, 1], \quad (1.13)$$

*and particularly,*

$$\begin{cases} C_1(1-x)^{\frac{1}{2}} \leq \rho_{sub}(x) - 1 \leq C_2(1-x)^{\frac{1}{2}}, \\ -C_3(1-x)^{-\frac{1}{2}} \leq \rho'_{sub}(x) \leq -C_4(1-x)^{-\frac{1}{2}}, \end{cases} \quad \text{for } x \text{ near } 1, \quad (1.14)$$

*where  $m = m(\tau, \underline{b}) < \bar{b} - 1$  is a small positive constant, and  $C_2 > C_1 > 0$  and  $C_3 > C_4 > 0$  are some positive constants;*

2. *At least one pair of supersonic solution  $(\rho_{sup}, E_{sup})(x) \in C^{\frac{1}{2}}[0, 1] \times H^1(0, 1)$  satisfying  $0 < \rho_{sup}(x) \leq 1$  and*

$$\begin{cases} C_5 x^{\frac{1}{2}} \leq 1 - \rho_{sup}(x) \leq C_6 x^{\frac{1}{2}}, \\ -C_7 x^{-\frac{1}{2}} \leq \rho'_{sup}(x) \leq -C_8 x^{-\frac{1}{2}}, \end{cases} \quad \text{for } x \text{ near } 0, \quad (1.15)$$

*where  $C_6 > C_5 > 0$  and  $C_7 > C_8 > 0$  are some positive constants.  $\rho_{sup}$  has only one critical point  $z_0$  over  $(0, 1)$ , such that  $(\rho_{sup})_x < 0$  on  $(0, z_0)$  and  $(\rho_{sup})_x > 0$  on  $(z_0, 1)$ .*

3. *Assume further that  $\tau$  is large and that  $\bar{b} - \underline{b} \ll 1$ , then equations (1.5)- (1.6) have infinitely many transonic solutions  $(\rho_{trans}, E_{trans})(x)$  combining stationary shocks which satisfy the entropy condition (1.10) and the Rankine-Hugoniot jump condition (1.11) at different jump locations  $x_0$ , where  $x_0$  can be uniquely determined when  $\rho_l$  satisfying  $\rho_r - \rho_l \ll 1$  is fixed, but the choice of  $\rho_l$  can be infinitely many;*
4. *Assume further that  $b(x) = b > 1$  is a constant, then when  $\tau$  is small enough, equations (1.5)- (1.6) have infinitely many  $C^1$  transonic solution; moreover, in this case there is no transonic shock solution.*

**Theorem 1.2** (Case of supersonic doping profile). *Let the doping profile be supersonic such that  $b(x) \in L^\infty(0, 1)$  and  $0 < b(x) \leq \bar{b} \leq 1$ . Then:*

1. *there is no interior subsonic solution to equations (1.5)- (1.6);*
2. *there is no interior supersonic solution nor transonic solution to (1.5)- (1.6), if the doping profile is sufficiently small such that  $\bar{b}(1 + \sqrt{2\bar{b}}) < 1$ ;*
3. *there is no interior supersonic solution nor transonic solution to (1.5)- (1.6), if the relaxation time is small with  $\tau < \frac{1}{3}$ ;*
4. *there exists at least one interior supersonic solution  $(\rho_{sup}, E_{sup})(x)$  to (1.5)- (1.6), satisfying  $\rho_{sup} \in C^{\frac{1}{2}}[0, 1]$  and the optimal estimate (1.15), if the doping profile  $b(x)$  is close to the sonic boundary  $\rho = 1$  and the relaxation time is sufficiently large  $\tau \gg 1$ ;*
5. *there exist infinitely many transonic shock solutions  $(\rho_{trans}, E_{trans})(x)$  to (1.5)- (1.6) joint with some stationary shocks satisfying the entropy condition (1.10) and the Rankine-Hugoniot jump condition (1.11) at different jump locations  $x_0$ , if the doping profile  $b(x)$  is close to the sonic boundary  $\rho = 1$  and the relaxation time is sufficiently large  $\tau \gg 1$ , where  $x_0$  can be uniquely determined when  $\rho_l$  satisfying  $\rho_r - \rho_l \ll 1$  is fixed, but the choice of  $\rho_l$  can be infinitely many.*

**Remark 1.1.** *In Parts 1 and 2 of Theorem 1.1, see also Part 4 of Theorem 1.2, the estimates (1.14) and (1.15) imply that  $C^{\frac{1}{2}}[0, 1]$  is the optimal Hölder space for the global regularity of the subsonic solution  $\rho_{sub}(x)$  and the supersonic solution  $\rho_{sup}(x)$ . Such a regularity  $C^{\frac{1}{2}}$  matches also the analysis for the steady nozzle in [28, 29].*

*In Part 3 of Theorem 1.1 and Part 5 of Theorem 1.2, when  $\tau \gg 1$ , namely, the semiconductor effect is small, then the steady hydrodynamic system possesses infinitely many transonic shock solutions. The similar results in [22, 21] can be regarded in some sense of our special example as  $\tau = \infty$ .*

*In Part 4 of Theorem 1.1, if  $b(x)$  is a constant and  $\tau$  is small, Part 4 implies that the regularity of the subsonic solution on the left boundary, as well as the regularity for the supersonic solution on the right boundary, can be lifted up to  $C^1$ . It seems that such a  $C^1$  regularity of transonic solutions is the first result obtained for semiconductor models so far. Essentially, the strong damping effect (the semiconductor effect) of  $-\frac{J}{\tau}$  makes the transonic solutions to be  $C^1$  smooth. Notice that the  $C^2$  transonic flow also arises in the finite de Laval nozzles, where the geometry structure causes the transonic flow to be smooth. For details, we refer to the interesting works of C. Wang and Z. Xin [30, 31].*



**Remark 1.2.** If  $b(x) \equiv 1$ , then the steady-state Euler-Poisson equations (1.5) and (1.6) admit the sonic solution  $(\rho_{sonic}, E_{sonic})(x) \equiv (1, \frac{1}{\tau})$ .

**Remark 1.3.** Theorem 1.2 indicates that when the doping profile is small enough, or the relaxation time is small enough, then the system has no solution. This also explains the physical phenomenon that the semiconductor device doesn't work efficiently when the background of the device is too pure.

The paper is organized as follows. In Section 2, we prove Theorem 1.1. The adopted approach is the method of viscosity vanishing and the technical energy method with the help of phase-plane analysis. The existence of infinity many  $C^1$ -smooth transonic solutions and transonic shocks are proved by the artful construction method. In Section 3, the main duty is to prove Theorem 1.2. Finally, in Section 4, when the pressure function is  $P(\rho) = T\rho^\gamma$  for  $\gamma > 1$ , the hydrodynamic system of Euler-Poisson equations becomes isentropic. We conclude that the results presented in Theorems 1.1 and 1.2 all hold for the isentropic system with  $\gamma > 1$ .

## 2 The case of subsonic doping profile

In this section, we assume that  $\underline{b} > 1$ . In other words, the doping profile is subsonic. First of all, let us test a special case when  $b(x) \equiv b > 1$  (constant), we may observe the structure of stationary solutions to system (1.5)-(1.6) from the phase-plane analysis. Notice that, when  $b > 1$ , the critical point of system (1.5) is  $A = (b, \frac{1}{\tau b})$ , and the Jacobian matrix of system (1.5) at  $A$  is:

$$J(A) = \begin{bmatrix} \frac{b}{\tau(b^2 - 1)} & \frac{b^3}{b^2 - 1} \\ 1 & 0 \end{bmatrix}.$$

It is easy to see that the eigenvalues  $\lambda$  of matrix  $J(A)$  satisfy the following characteristic equation

$$\lambda^2 - \frac{b\lambda}{\tau(b^2 - 1)} - \frac{b^3}{b^2 - 1} = 0. \quad (2.1)$$

Notice that,  $\lambda_1 \lambda_2 = -\frac{b^3}{b^2 - 1} < 0$ , where  $\lambda_1$  and  $\lambda_2$  are the roots of (2.1). Thus,  $A$  is a saddle point. On the other hand, it follows from system (1.5) that

$$\frac{d\mathbf{E}}{d\rho} = \frac{(\rho - b)(1 - \frac{1}{\rho^2})}{\rho\mathbf{E} - \frac{1}{\tau}}, \quad (2.2)$$

which helps to determine the directions of all trajectories. Here and in the sequel, to avoid confusion, we denote by  $\mathbf{E} = \mathbf{E}(\rho)$  the function of the trajectory.

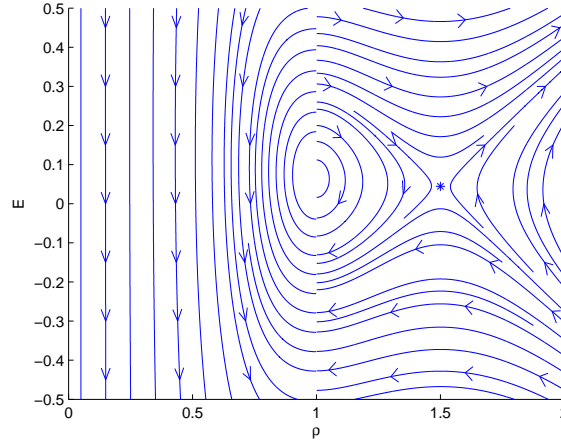


Figure 1: Phase plane of  $(\rho, E)$  with  $\tau = 15$  and  $b = 1.5$ ; \* is the saddle point  $A = (1.5, 2/45)$ .

Figure 1 is the phase-plane of  $(\rho, \mathbf{E})$  with  $\tau = 15$  and  $b = 1.5$ , from which we observe that there exist at least one interior subsonic solution and one interior supersonic solution. In Figure 2, we draw the profiles of the interior subsonic solution and interior supersonic solution. Figure 3 demonstrates how to construct an interior transonic shock solution when  $\tau$  is large: the discontinuous trajectory in blue stands for a transonic shock solution with smaller length (e.g.  $\frac{1}{2}$ ) and is structured by a stationary shock at  $x_0$  with the Rankine-Hugoniot jump condition (1.11) linking the other two solutions: one is a supersonic solution  $\rho_{sup}(x)$  with  $\rho_{sup}(0) = 1$  and  $\rho_{sup}(x_0^-) = \rho_l < 1$ , and the other is a subsonic solution  $\rho_{sub}(x)$  with  $\rho_{sub}(x_0^+) = \rho_r > 1$  and  $\rho_{sub}(\frac{1}{2}) = 1$ ; the discontinuous trajectory in red represents a similar transonic shock solution with larger length (e.g.  $\frac{3}{2}$ ) satisfying the entropy condition and the Rankine-Hugoniot condition at some jump location. By continuity, there is an interior transonic shock solution to (1.5) on  $[0, 1]$ . Since the choice of  $\rho_l = \rho_{sup}(x_0^+)$  can be infinitely many when  $\rho_r - \rho_l \ll 1$ , there are infinitely many transonic shock solutions. In Figure 4, we draw two transonic shock solutions to system (1.5) with different  $\rho_l$ .

While, when  $\tau$  is small, we see in Figure 5 that the phase-plane changes dramatically: many subsonic trajectories start from the same point  $(1, \frac{1}{\tau})$ , and many supersonic trajectories end at the same point  $(1, \frac{1}{\tau})$ . As a result, one can see that there are possibly smooth transonic solutions, which is constructed by two solutions at some location  $x_0$ : one is an interior supersonic solution with  $\rho_{sup}(0) = 1 = \rho_{sup}(x_0)$ , and the other is an interior subsonic solution with  $\rho_{sub}(x_0) = 1 = \rho_{sub}(1)$ . Since the transition location  $x_0$  can be chosen arbitrarily in  $(0, 1)$ , these smooth transonic solutions are infinitely many.

Next we are going to prove Theorem 1.1 for a subsonic doping profile  $b(x) > 1$  in general

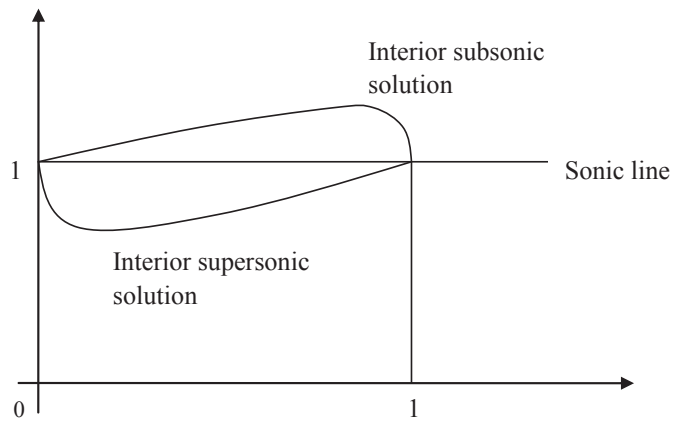


Figure 2: Interior subsonic solution and interior supersonic solution for the case of subsonic doping profile.

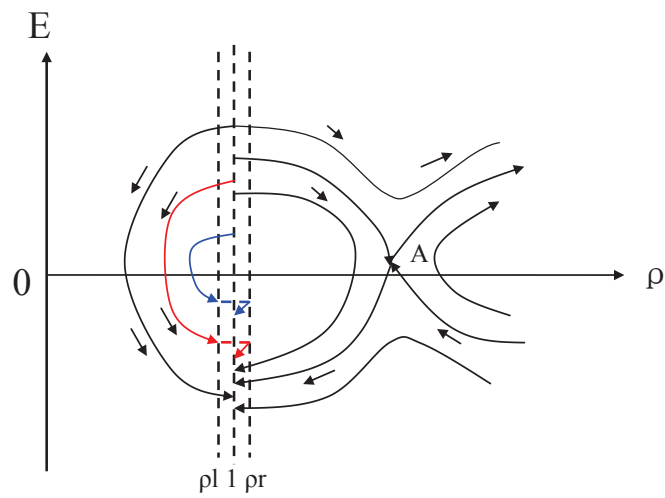


Figure 3: Transonic shock trajectories in the phase plane of  $(\rho, E)$  for the case of subsonic doping profile when  $\tau$  is large.

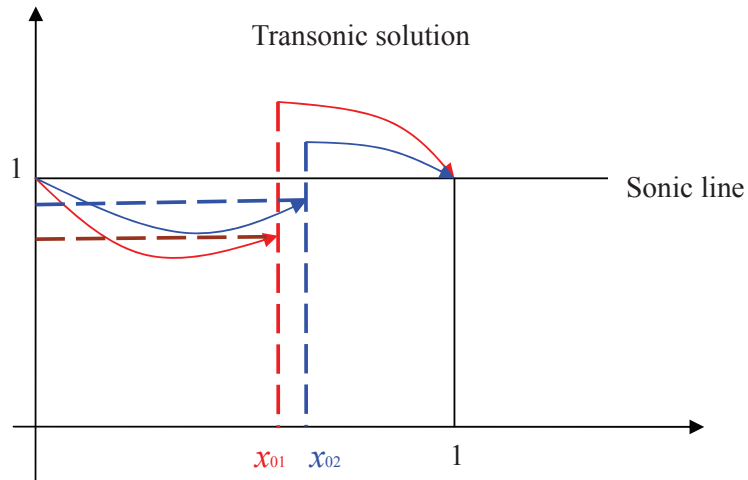


Figure 4: Transonic shock solutions in the case of subsonic doping profile when  $\tau$  is large.

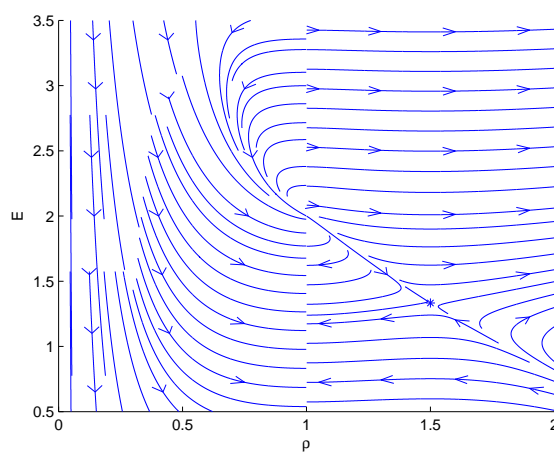


Figure 5: Phase plane of  $(\rho, E)$  with  $\tau = 0.5$  and  $b = 1.5$ ; \* is the saddle point  $A = (1.5, 4/3)$ .

form.

## 2.1 Unique interior subsonic solution

Firstly, we prove that there exists a unique interior subsonic solution to equation (1.7). The adopted approach is the technical compactness method, which is inspired by the vanishing viscosity method.

**Theorem 2.1.** *Assume that  $b \in L^\infty(0, 1)$  and  $\underline{b} > 1$ , then equation (1.7) has a unique interior subsonic solution  $\rho_{sub}$  satisfying*

$$1 + m \sin(\pi x) \leq \rho_{sub} \leq \bar{b}, \quad x \in [0, 1], \quad (2.3)$$

where  $m = m(\tau, \underline{b})$  is a positive constant.

Since the equation (1.7) is partially elliptic but degenerates at the boundary, so the corresponding solution to (1.7) will lack the necessary regularity, and we cannot directly work on (1.7). In order to prove Theorem 2.1, now we consider the following approximate equation:

$$\begin{cases} \left[ \left( \frac{1}{\rho_j} - \frac{j^2}{(\rho_j)^3} \right) (\rho_j)_x \right]_x + \left( \frac{j}{\tau \rho_j} \right)_x - [\rho_j - b(x)] = 0, & x \in (0, 1) \\ \rho_j(0) = \rho_j(1) = 1, \end{cases} \quad (2.4)$$

where the parameter  $j$  is chosen as a constant such that  $0 < j < 1$ . Thus, the equation (2.4) is expected to be uniformly elliptic in  $[0, 1]$ , because  $\frac{1}{\rho_j} - \frac{j^2}{\rho_j^3} = \frac{1}{\rho_j^3}(\rho_j + j)(\rho_j - j) > 0$  for the expected solution  $\rho_j \geq 1$ . To show the wellposedness of the approximate equation (2.4) and to establish the lower bound estimate in (2.3), we need the following comparison principle.

**Lemma 2.1** (Comparison principle). *Let  $U \in C^1[0, 1]$  be a weak solution of (2.4) satisfying  $U \geq 1$  on  $[0, 1]$ , and that*

$$\int_0^1 \left[ \left( \frac{1}{U} - \frac{j^2}{U^3} \right) U_x + \frac{j}{\tau U} \right] \varphi_x dx + \int_0^1 (U - b) \varphi dx = 0 \quad \text{for any } \varphi \in H_0^1(0, 1), \quad (2.5)$$

where  $0 < j < 1$  is a constant, and let  $V \in C^1[0, 1]$  be such that  $V(x) > 0$  for  $x \in [0, 1]$ ,  $V(0) \leq 1$ ,  $V(1) \leq 1$  and

$$\int_0^1 \left[ \left( \frac{1}{V} - \frac{j^2}{V^3} \right) V_x + \frac{j}{\tau V} \right] \varphi_x dx + \int_0^1 (V - b) \varphi dx \leq 0 \quad \text{for any } \varphi \geq 0, \quad \varphi \in H_0^1(0, 1).$$

Then  $U(x) \geq V(x)$  over  $[0, 1]$ .

*Proof.* Inspired by the textbook [14] (see Theorem 2.7 in Section 10.4), we can prove this comparison principle. Let us denote

$$A(z, p) := \left( \frac{1}{z} - \frac{j^2}{z^3} \right) p + \frac{j}{\tau z}$$

for simplicity. Then, for any  $\varphi \in H_0^1(0, 1)$ ,  $\varphi \geq 0$ , we have

$$\int_0^1 [A(V, V_x) - A(U, U_x)] \varphi_x dx + \int_0^1 (V - U) \varphi dx \leq 0. \quad (2.6)$$

Set  $e(x) := V(x) - U(x)$ . A simple calculation gives

$$\begin{aligned} A(V, V_x) - A(U, U_x) &= A(V, V_x) - A(U, V_x) + A(U, V_x) - A(U, U_x) \\ &= \int_0^1 \frac{\partial A}{\partial z}(V_t, V_x) dt \cdot e(x) + \int_0^1 \frac{\partial A}{\partial p}(U, (V_t)_x) dt \cdot e_x(x), \end{aligned}$$

where  $V_t(x) := tV(x) + (1-t)U(x)$ . Taking  $\varphi(x) = \frac{e^+(x)}{e^+(x)+h}$  with  $e^+(x) := \max\{0, e(x)\}$  and  $h > 0$  being a constant, a straightforward computation yields

$$[\ln(1 + e^+(x)/h)]_x = \frac{e_x^+(x)}{e^+(x) + h} \quad \text{and} \quad \varphi_x = \frac{h}{e^+(x) + h} [\ln(1 + e^+(x)/h)]_x.$$

Since  $0 < j < 1$ ,  $v \in C^1[0, 1]$  and  $\min_{x \in [0, 1]} v > 0$ , it is easy to see that

$$\begin{aligned} \int_0^1 \frac{\partial A}{\partial p}(U, (V_t)_x) dt &= \frac{1}{U} - \frac{1}{U^3} + \frac{1-j^2}{U^3} \geq \frac{1-j^2}{\|U\|_{L^\infty}^3}, \\ \int_0^1 \frac{\partial A}{\partial z}(V_t, V_x) dt &\leq C\|V_x\|_{C[0, 1]} + \frac{Cj}{\tau} \leq C. \end{aligned}$$

It then follows from (2.6) that

$$\begin{aligned} &\frac{h(1-j^2)}{\|U\|_{L^\infty}^3} \int_0^1 |[\ln(1 + e^+(x)/h)]_x|^2 dx + \int_0^1 \frac{(e^+(x))^2}{e^+(x) + h} dx \\ &\leq Ch \int_0^1 \frac{e^+(x)}{e^+(x) + h} |[\ln(1 + e^+(x)/h)]_x| dx \\ &\leq \frac{h(1-j^2)}{2\|U\|_{L^\infty}^3} \int_0^1 |[\ln(1 + e^+(x)/h)]_x|^2 dx + \frac{C^2 h \|U\|_{L^\infty}^3}{2(1-j^2)}, \end{aligned}$$

where we have used Young's inequality in the second inequality. Thus,

$$\int_0^1 |[\ln(1 + e^+(x)/h)]_x|^2 dx \leq \frac{C^2 \|U\|_{L^\infty}^6}{(1-j^2)^2} \quad \text{for any } h > 0.$$

This inequality together with Poincaré's inequality leads to

$$\int_0^1 [\ln(1 + e^+(x)/h)]^2 dx \leq 2 \int_0^1 |[\ln(1 + e^+(x)/h)]_x|^2 dx \leq \frac{C^2 \|U\|_{L^\infty}^6}{(1-j^2)^2} \quad \text{for any } h > 0. \quad (2.7)$$

Now letting  $h \rightarrow 0^+$ , one can see that if  $e^+(x) \neq 0$  for some  $x \in (0, 1)$ , then

$$\lim_{h \rightarrow 0^+} \int_0^1 |\ln(1 + e^+(x)/h)|^2 dx = \infty,$$

which is a contradiction to (2.7). Therefore,  $U(x) \geq V(x)$  over  $[0, 1]$ .  $\square$

Let us now prove the wellposedness of equation (2.4).

**Lemma 2.2.** *Assume that  $b(x) \in L^\infty(0, 1)$  and  $\underline{b} > 1$ , then (2.4) admits a unique weak solution  $\rho_j$  satisfying  $\rho_j \in H_0^1(0, 1)$  and*

$$1 + m \sin(\pi x) \leq \rho_j(x) \leq \bar{b}, \quad x \in [0, 1], \quad (2.8)$$

where  $m = m(\tau, \underline{b}) < \bar{b} - 1$  is a positive constant independent of  $j$ .

**Remark 2.1.** *In [9], Degond and Markowich also obtained the uniqueness of the subsonic solution, but they needed to restrict the current density sufficiently small  $j \ll 1$  (the completely subsonic case). Here, we still have the uniqueness of the subsonic solution for any  $j$  with  $0 < j < 1$  in the case of subsonic doping profile.*

*Proof.* Because  $0 < j < 1$ , the fluid velocity of equation (2.4) is  $j/\rho_j$ , which is subsonic if  $\rho_j \geq 1$ . In other words, equation (2.4) is uniformly elliptic for  $\rho_j \geq 1$ . Recall Theorem 1 of [9], equation (2.4) has a subsonic weak solution  $\rho_j \in H^2(0, 1)$  satisfying  $1 \leq \rho_j(x) \leq \bar{b}$ . Thus, we only need to show that such  $\rho_j$  is unique for any  $0 < j < 1$ , and to establish the lower bound estimate in (2.8).

Suppose that there are two solutions  $u$  and  $v$  satisfying  $u, v \geq 1$ ,  $u, v \in H^2(0, 1)$ . By the Sobolev imbedding theorem,  $u, v \in C^1[0, 1]$ . Hence, the comparison principle (Lemma 2.1) gives  $u(x) = v(x)$  over  $[0, 1]$ .

We now derive the lower bound estimate for  $\rho_j(x)$ . Denote

$$q(x) := 1 + m \sin(\pi x),$$

where  $m > 0$  is a constant to be determined later. Since  $0 < j < 1$ , it is easy to calculate that

$$-\left[\left(\frac{1}{q} - \frac{j^2}{q^3}\right) q_x\right]_x - \left(\frac{j}{\tau q}\right)_x + (q - b) \leq Cm + (1 - b) \leq Cm + (1 - \underline{b}) < 0,$$

if  $m$  is small enough such that  $Cm < (\underline{b} - 1)$ . Here  $C = C(\tau)$  is a positive constant. Thus, by Lemma 2.1 again, we have  $\rho_j(x) \geq q(x) = 1 + m \sin(\pi x)$  on  $[0, 1]$ .  $\square$

*Proof of Theorem 2.1.* Multiplying (2.4) by  $(\rho_j - 1)$ , we have

$$\begin{aligned} (1 - j^2) \int_0^1 \frac{|(\rho_j)_x|^2}{(\rho_j)^3} dx + \frac{4}{9} \int_0^1 \frac{(\rho_j + 1)}{(\rho_j)^3} \cdot |((\rho_j - 1)^{\frac{3}{2}})_x|^2 dx \\ + \frac{j}{\tau} \int_0^1 \frac{(\rho_j)_x}{\rho_j} dx + \int_0^1 (\rho_j - b)(\rho_j - 1) dx = 0. \end{aligned} \quad (2.9)$$

Noting that

$$\begin{aligned} \frac{j}{\tau} \int_0^1 \frac{(\rho_j)_x}{\rho_j} dx &= \frac{j}{\tau} \int_0^1 (\ln \rho_j)_x dx = 0, \\ \int_0^1 (\rho_j - b)(\rho_j - 1) dx &= \int_0^1 (\rho_j - 1)^2 dx + \int_0^1 (1 - b)(\rho_j - 1) dx \\ &\geq \frac{1}{2} \int_0^1 (\rho_j - 1)^2 dx - \frac{1}{2} \int_0^1 (b - 1)^2 dx, \end{aligned}$$

$0 < j < 1$ , and  $1 \leq \rho_j \leq \bar{b}$ , it follows from (2.9) that

$$\begin{aligned} \frac{(1 - j^2)}{\bar{b}^3} \int_0^1 |(\rho_j)_x|^2 dx + \frac{8}{9\bar{b}^3} \int_0^1 \left| ((\rho_j - 1)^{\frac{3}{2}})_x \right|^2 dx + \frac{1}{2} \int_0^1 (\rho_j - 1)^2 dx \\ \leq \frac{1}{2} \int_0^1 [b(x) - 1]^2 dx, \end{aligned}$$

which gives

$$\left\| (\rho_j - 1)^{\frac{3}{2}} \right\|_{H^1} \leq C \text{ and } \left\| (1 - j^2)(\rho_j)_x \right\|_{L^2} \leq C(1 - j^2)^{\frac{1}{2}}. \quad (2.10)$$

Thus, by the compact imbedding  $H^1(0, 1) \hookrightarrow C^{1/2}[0, 1]$ , there exists a function  $\rho$  such that, as  $j \rightarrow 1^-$ , up to a subsequence,

$$(\rho_j - 1)^{\frac{3}{2}} \rightharpoonup (\rho - 1)^{\frac{3}{2}} \text{ weakly in } H^1(0, 1), \quad (2.11)$$

$$(\rho_j - 1)^{\frac{3}{2}} \rightarrow (\rho - 1)^{\frac{3}{2}} \text{ strongly in } C^{\frac{1}{2}}[0, 1], \quad (2.12)$$

$$(1 - j^2)(\rho_j)_x \rightarrow 0 \text{ strongly in } L^2(0, 1). \quad (2.13)$$

Observing that  $((\rho_j - 1)^2)_x = \frac{4}{3}(\rho_j - 1)^{\frac{1}{2}}((\rho_j - 1)^{\frac{3}{2}})_x$ , we get from (2.10) that

$$\left\| (\rho_j - 1)^2 \right\|_{H^1} = \left\| (\rho_j - 1)^2 \right\|_{L^2} + \left\| ((\rho_j - 1)^2)_x \right\|_{L^2} \leq C \left\| (\rho_j - 1)^{\frac{3}{2}} \right\|_{H^1} \leq C,$$

which leads to

$$(\rho_j - 1)^2 \rightharpoonup (\rho - 1)^2 \text{ weakly in } H^1(0, 1) \text{ as } j \rightarrow 1^-. \quad (2.14)$$

Now we multiply (2.4) by  $\varphi \in H_0^1(0, 1)$  to derive

$$\begin{aligned} \frac{1}{2} \int_0^1 \frac{\rho_j + 1}{\rho_j^3} [(\rho_j - 1)^2]_x \varphi_x dx + \int_0^1 \frac{1}{\rho_j^3} (1 - j^2)(\rho_j)_x \varphi_x dx \\ + \frac{j}{\tau} \int_0^1 \frac{\varphi_x}{\rho_j} dx + \int_0^1 [\rho_j(x) - b(x)] \varphi dx = 0. \end{aligned}$$



Letting  $j \rightarrow 1^-$ , and applying (2.12)-(2.14), we prove the existence of weak solution  $\rho(x) = \rho_{sub}(x)$  satisfying (1.8). Since  $m$  presented in (2.8) is independent of  $j$ , then the lower bound estimate in (2.3) immediately follows from (2.8) and (2.12).

To prove the uniqueness of interior subsonic solution, we first need to investigate the regularity of  $w(x)$  defined by  $w(x) := (\rho(x) - 1)^2$ . Clearly,  $w \in H_0^1(0, 1)$ . From (1.7), it can be verified that  $w$  satisfies

$$\left( \frac{(2 + \sqrt{w})w_x}{2(1 + \sqrt{w})^3} + \frac{1}{\tau(1 + \sqrt{w})} \right)_x - (\sqrt{w} + 1 - b) = 0, \quad x \in (0, 1). \quad (2.15)$$

For simplicity, we set

$$f_1(x) := \frac{2 + \sqrt{w}}{(1 + \sqrt{w})^3}, \quad f_2(x) := \frac{1}{1 + \sqrt{w}}, \quad f_3(x) := \frac{f_1(x)w_x(x)}{2} + \frac{f_2(x)}{\tau}.$$

Because (2.15) holds in the sense of distribution, we have  $f_3 \in H^1(0, 1)$ . By Sobolev imbedding theorem, we have  $w, f_3 \in C^{1/2}[0, 1]$ . Since  $w \geq 0$  on  $[0, 1]$ , then

$$|\sqrt{w(y)} - \sqrt{w(x)}| = \frac{|w(y) - w(x)|}{\sqrt{w(y)} + \sqrt{w(x)}} \leq \frac{|w(y) - w(x)|}{\sqrt{|w(y) - w(x)|}} \leq C|y - x|^{1/4}.$$

On the other hand, for any  $x, y \in [0, 1]$ , it holds

$$f_2(x) - f_2(y) = \frac{1}{1 + \sqrt{w(x)}} - \frac{1}{1 + \sqrt{w(y)}} = \frac{\sqrt{w(y)} - \sqrt{w(x)}}{(1 + \sqrt{w(x)})(1 + \sqrt{w(y)})}.$$

Thus,

$$|f_2(x) - f_2(y)| \leq |\sqrt{w(y)} - \sqrt{w(x)}| \leq C|y - x|^{1/4}.$$

This means  $f_2 \in C^{1/4}[0, 1]$ . Similarly, we have  $f_1 \in C^{1/4}[0, 1]$ . Notice that  $w_x = \frac{2f_3 - 2f_2/\tau}{f_1} \in C^{1/4}[0, 1]$ , then

$$w \in C^{1+1/4}[0, 1]. \quad (2.16)$$

Now, integrating (2.15) over  $[0, x]$  and setting  $G_w(x) := \frac{(2 + \sqrt{w(x)})w_x(x)}{2(1 + \sqrt{w(x)})^3} + \frac{1}{\tau(1 + \sqrt{w(x)})}$ , then

$$\begin{cases} \frac{(2 + \sqrt{w})w_x}{2(1 + \sqrt{w})^3} = G_w - \frac{1}{\tau(1 + \sqrt{w})}, \\ G_w(x) = G_w(0) + \int_0^x [\sqrt{w(s)} + 1 - b(s)]ds. \end{cases} \quad (2.17)$$

We are now ready to prove the uniqueness of interior subsonic solution. Suppose  $\rho_1(x)$  and  $\rho_2(x)$  are two different interior subsonic solutions to equation (1.7). So, there exists at least a number  $z \in (0, 1)$  such that  $\rho_1(z) \neq \rho_2(z)$ . Without loss of generality, we may assume  $\rho_1(z) > \rho_2(z)$ , then  $w_1(z) > w_2(z)$ . Since  $w_1, w_2 \in C^{1+1/4}[0, 1]$ , there exists a maximal interval  $[a, c] \subset [0, 1]$  such that  $z \in (a, c)$ ,

$$w_1(a) = w_2(a), \quad w_1(c) = w_2(c) \quad \text{and} \quad w_1(x) > w_2(x), \quad x \in (a, c).$$

Obviously, it holds

$$(w_1)_x(a) = \lim_{x \rightarrow a^+} \frac{w_1(x) - w_1(a)}{x - a} \geq \lim_{x \rightarrow a^+} \frac{w_2(x) - w_2(a)}{x - a} = (w_2)_x(a), \quad (2.18)$$

$$(w_1)_x(c) = \lim_{x \rightarrow c^-} \frac{w_1(x) - w_1(c)}{x - c} \leq \lim_{x \rightarrow c^-} \frac{w_2(x) - w_2(c)}{x - c} = (w_2)_x(c). \quad (2.19)$$

Owing to (2.19) and the first equation of (2.17),

$$G_{w_1}(c) \leq G_{w_2}(c).$$

Substituting this inequality into the second equation of (2.17), we have

$$G_{w_1}(a) + \int_a^c [\sqrt{w_1(x)} + 1 - b(x)] dx \leq G_{w_2}(a) + \int_a^c [\sqrt{w_2(x)} + 1 - b(x)] dx.$$

Since  $w_1(x) > w_2(x)$  over  $(a, c)$ , then

$$G_{w_1}(a) < G_{w_2}(a).$$

Using the first equation of (2.17) again, we obtain

$$(w_1)_x(a) < (w_2)_x(a),$$

which contradicts to (2.18). Therefore,  $\rho_1(x) = \rho_2(x)$  over  $[0, 1]$ , namely, the interior subsonic solution  $\rho_{sub}(x)$  is unique.  $\square$

We proceed to study the regularity of this interior subsonic solution.

**Proposition 2.1.**  $\rho_{sub} \in C^{1/2}[0, 1]$ , and there exist  $0 < s_1 < 1$ ,  $C_i$  ( $i = 1, 2, 3, 4$ ) such that

$$\begin{aligned} C_1(1-x)^{1/2} &< \rho_{sub}(x) - 1 < C_2(1-x)^{1/2}, \\ -C_3(1-x)^{-1/2} &< (\rho_{sub})_x(x) < -C_4(1-x)^{-1/2}, \end{aligned} \quad \text{for } x \in [1-s_1, 1]. \quad (2.20)$$

**Remark 2.2.** This proposition indicates that  $\frac{1}{2}$  is the optimal exponent in Hölder space for the global regularity of the unique interior subsonic solution  $\rho_{sub}(x)$ . And the derivative of the approximate subsonic solution sequence  $\{\rho_j\}_{0 < j < 1}$  constructed in Lemma 2.2 blows up as  $j \rightarrow 1^-$  for  $x \approx 1$ , namely,  $\lim_{j \rightarrow 1^-} \rho'_j(x) = -\infty$  for  $x \approx 1$ .

*Proof.* For convenience, we denote by  $\rho$  the interior subsonic solution of (1.5). By (2.16), we have  $(\rho - 1)^2 = w \in C^1[0, 1]$ . Since  $\rho \geq 1$  on  $[0, 1]$ , then

$$|\rho(x) - 1 + \rho(y) - 1| = |\rho(x) - 1| + |\rho(y) - 1| \geq |(\rho(x) - 1) - (\rho(y) - 1)| = |\rho(x) - \rho(y)|.$$

Thus, we have

$$\frac{|\rho(x) - \rho(y)|^2}{|x - y|} = \frac{|\rho(x) - \rho(y)| |(\rho(x) - 1)^2 - (\rho(y) - 1)^2|}{|x - y| |\rho(x) - 1 + \rho(y) - 1|} \leq \frac{|w(x) - w(y)|}{|x - y|} \leq C,$$

for any  $x, y \in [0, 1]$ , which indicates that  $\rho \in C^{1/2}[0, 1]$ .

Now we are going to prove the estimates in (2.20). We first claim  $E(1) < \frac{1}{\tau}$ . Otherwise, if  $E(1) \geq \frac{1}{\tau}$ , then it will imply a contradiction. In fact, since  $\rho \in C[0, 1]$  and  $\rho(1) = 1 < \underline{b} \leq b(x)$  for  $x \in [0, 1]$ , there exists  $\hat{\epsilon} > 0$  such that  $\rho(x) - b(x) < 0$  for a.e.  $x \in [1 - \hat{\epsilon}, 1]$ . By integrating the second equation of (1.5) over  $[x, 1]$  for  $x \in [1 - \hat{\epsilon}, 1]$ , we have

$$E(x) = E(1) - \int_x^1 [\rho(s) - b(s)] ds > E(1) \geq \frac{1}{\tau}, \quad \text{for } x \in [1 - \hat{\epsilon}, 1].$$

Noting  $\rho(x) > 1$  over  $(0, 1)$ , we have  $E(x) - \frac{1}{\tau \rho(x)} \geq \frac{1}{\tau} \left(1 - \frac{1}{\rho(x)}\right) > 0$  for  $x \in [1 - \hat{\epsilon}, 1]$ . It then follows from the first equation of (1.5) that  $\rho_x(x) > 0$  on  $[1 - \hat{\epsilon}, 1]$ , which contradicts to the fact that  $\rho(1) = 1$  and  $\rho(x) > 1$  over  $(0, 1)$ .

Now let  $q := E(1) - \frac{1}{\tau}$ , then  $q < 0$ . Based on the continuity of the function  $\left(E(x) - \frac{1}{\tau \rho(x)}\right)$ , there exists a number  $0 < s_1 < \hat{\epsilon}$  such that

$$\frac{3q}{2} \leq E(x) - \frac{1}{\tau \rho(x)} \leq \frac{q}{2} < 0 \quad \text{for } x \in [1 - s_1, 1]. \quad (2.21)$$

From the first equation of (1.5), we have

$$E(x) - \frac{1}{\tau \rho(x)} = \left(1 - \frac{1}{\rho^2}\right) \frac{\rho_x}{\rho} = \frac{\rho + 1}{\rho^3} (\rho - 1) \rho_x = \frac{\rho + 1}{2\rho^3} \left((\rho - 1)^2\right)_x.$$

Applying (2.21) to the above equation, we then have

$$\frac{3q\rho^3(x)}{\rho(x) + 1} \leq \left((\rho - 1)^2\right)_x = \left[E(x) - \frac{1}{\tau \rho(x)}\right] \frac{2\rho^3(x)}{\rho(x) + 1} \leq \frac{q\rho^3(x)}{\rho(x) + 1} < 0 \quad \text{for } x \in [1 - s_1, 1].$$

Applying (2.3) to the above inequalities, we can estimate

$$\frac{3q\bar{b}^3}{2} < \left((\rho(x) - 1)^2\right)_x < \frac{q}{\bar{b} + 1} < 0 \quad \text{for } x \in [1 - s_1, 1]. \quad (2.22)$$

Integrating (2.22) over  $[x, 1]$  for  $x \in [1 - s_1, 1]$ , we get

$$C_1(1 - x)^{\frac{1}{2}} < \rho(x) - 1 < C_2(1 - x)^{\frac{1}{2}}, \quad \text{for } x \in [1 - s_1, 1], \quad (2.23)$$

with

$$C_1 := \sqrt{\frac{|q|}{\bar{b} + 1}} \quad \text{and} \quad C_2 := \sqrt{\frac{3|q|\bar{b}^3}{2}}.$$

Furthermore, from (2.22), we have

$$\frac{3q\bar{b}^3}{4(\rho(x) - 1)} < \rho_x(x) < \frac{q}{2(\bar{b} + 1)(\rho(x) - 1)} < 0 \text{ for } x \in [1 - s_1, 1].$$

This with (2.23) together implies

$$-C_3(1 - x)^{-\frac{1}{2}} < \rho_x(x) < -C_4(1 - x)^{-\frac{1}{2}}, \quad x \in [1 - s_1, 1],$$

for some positive constants  $C_3$  and  $C_4$ . The proof is complete.  $\square$

## 2.2 Interior supersonic solutions

We next prove the existence of interior supersonic solutions of (1.7).

**Theorem 2.2.** *Assume that  $b \in L^\infty(0, 1)$  and  $\underline{b} > 1$ , then equation (1.7) admits an interior supersonic solution  $\rho_{sup}(x)$  satisfying  $\ell \leq \rho_{sup}(x) \leq 1$  over  $[0, 1]$  for some positive constant  $\ell$ . Moreover,  $\rho_{sup}$  satisfies the following properties.*

- (i) *For any  $\frac{1}{2} > \epsilon > 0$ , there exists a number  $\delta > 0$  such that  $\rho_{sup}(x) \leq 1 - \delta$  for any  $x \in [\epsilon, 1 - \epsilon]$ .*
- (ii)  *$\rho_{sup}$  has only one critical point  $z_0$  over  $(0, 1)$  such that  $(\rho_{sup})_x < 0$  on  $(0, z_0)$  and  $(\rho_{sup})_x > 0$  on  $(z_0, 1)$ , i.e.  $z_0$  is the minimal point.*

As shown in the proof of Theorem 2.1, we consider the approximate equation

$$\begin{cases} \left[ \left( \frac{1}{\rho_k} - \frac{k^2}{(\rho_k)^3} \right) (\rho_k)_x \right]_x + \left( \frac{k}{\tau \rho_k} \right)_x - [\rho_k(x) - b(x)] = 0, & x \in (0, 1) \\ \rho_k(0) = \rho_k(1) = 1, \end{cases} \quad (2.24)$$

but with the parameter  $1 < k < \infty$ .

**Lemma 2.3.** *Let the doping profile be subsonic with  $b(x) \in L^\infty(0, 1)$  and  $\underline{b} > 1$ . Then (2.24) admits a weak solution  $\rho_k(x)$  satisfying*

$$\rho_k \in H^1(0, 1) \text{ and } 0 < \rho_k(x) \leq 1 \text{ over } [0, 1]. \quad (2.25)$$

**Remark 2.3.** *Peng and Violet [25] showed that if  $k$  is large enough, then equation (2.24) has a supersonic solution. Our Lemma 2.3 further show that, in the case of subsonic doping profile, for all  $1 < k < \infty$ , equation (2.24) has a supersonic solution. So, our result essentially improves the previous study in [25].*

*Proof.* The velocity  $u_k(x) = \frac{k}{\rho_k(x)}$  satisfies

$$\begin{cases} \left[ \left( u_k - \frac{1}{u_k} \right) (u_k)_x \right]_x + \frac{(u_k)_x}{\tau} - \left( \frac{k}{u_k} - b \right) = 0, & x \in (0, 1) \\ u_k(0) = u_k(1) = k. \end{cases} \quad (2.26)$$

So we only need to show that (2.26) has a weak solution  $u_k \in H^1(0, 1)$  satisfying  $k \leq u_k < \infty$ . To this end, we define an operator  $\mathcal{T} : \psi \rightarrow u$  by solving the following linear elliptic equation

$$\begin{cases} \left[ \left( \psi - \frac{1}{\psi} \right) u_x \right]_x + \frac{u_x}{\tau} - \left( \frac{k}{\psi} - b \right) = 0, & x \in (0, 1) \\ u(0) = u(1) = k. \end{cases} \quad (2.27)$$

Set

$$\mathcal{X} := \{ \psi(x) : \psi \in C^1[0, 1], k \leq \psi(x) \leq M, \psi(0) = \psi(1) = k, \|\psi\|_{C^\alpha[0, 1]} \leq \Lambda, \|\psi\|_{C^1[0, 1]} \leq \Upsilon(\Lambda) \},$$

where  $0 < \alpha < 1/2$ ,  $M$ ,  $\Lambda$  and  $\Upsilon(\Lambda)$  are some positive constants to be determined later. Suppose that  $\psi \in \mathcal{X}$ , by  $L^2$  theory of elliptic equation and the Sobolev imbedding theorem, we see that equation (2.27) has a unique solution  $u \in C^{1+\alpha}[0, 1]$  for  $0 < \alpha < 1$ . Multiplying (2.27) by  $(u - k)^-(x) := \min\{0, (u - k)(x)\}$ , we have

$$\int_0^1 \left( \psi - \frac{1}{\psi} \right) |[(u - k)^-]_x|^2 dx - \frac{1}{\tau} \int_0^1 u_x (u - k)^- dx + \int_0^1 \left( \frac{k}{\psi} - b \right) (u - k)^- dx = 0. \quad (2.28)$$

Because  $k > 1$  and  $\psi \geq k$ , we have  $\psi - \frac{1}{\psi} \geq k - 1 > 0$ , and noting that

$$\frac{1}{\tau} \int_0^1 u_x (u - k)^- dx = \frac{1}{2\tau} \int_0^1 ([(u - k)^-]^2)_x dx = 0,$$

it follows from (2.28) that

$$(k - 1) \int_0^1 |[(u - k)^-]_x|^2 dx + \int_0^1 \left( \frac{k}{\psi} - b \right) (u - k)^- dx \leq 0. \quad (2.29)$$

This inequality in combination with the fact that  $\frac{k}{\psi(x)} - b(x) < 0$  gives  $(u - k)^-(x) = 0$  for all  $x \in [0, 1]$ . Thus,  $u(x) \geq k$  over  $[0, 1]$ . Now multiplying (2.27) by  $(u - k)$ , just as shown in (2.29), using Young's inequality and Poincaré's inequality, we get

$$\begin{aligned} (k - 1) \int_0^1 |(u - k)_x|^2 dx &\leq \int_0^1 \left( b - \frac{1}{\psi} \right) (u - k) dx \\ &\leq \frac{k - 1}{2} \int_0^1 (u - k)^2 dx + \frac{1}{2(k - 1)} \int_0^1 \left( b(x) - \frac{k}{\psi} \right)^2 dx \\ &\leq \frac{k - 1}{2} \int_0^1 |(u - k)_x|^2 dx + \frac{1}{2(k - 1)} \int_0^1 b^2(x) dx. \end{aligned}$$

It then follows that

$$\|u_x\|_{L^2(0, 1)} \leq \frac{\|b\|_{L^2}}{k - 1}.$$

Furthermore, a straightforward computation yields

$$0 < u(x) \leq k + \frac{\|b\|_{L^2}}{k - 1}.$$

Thus, the compact imbedding of  $H^1(0, 1)$  into  $C^{\alpha_0}[0, 1]$  with  $0 < \alpha_0 < 1/2$  gives

$$\|u\|_{C^{\alpha_0}[0,1]} \leq C_0(k, \|b\|_{L^2}) \text{ for a constant } C_0 > 0.$$

Hence we determine  $M = 1 + \frac{\|b\|_{L^2}^2}{k-1}$ ,  $\alpha = \alpha_0$  and  $\Lambda = C_0(k, \|b\|_{L^2})$ . By the Hölder estimate for the first order derivative of divergence form elliptic equation [14], we derive

$$\|u\|_{C^{1+\alpha}[0,1]} \leq C_1(k, \|b\|_{L^2}, \Lambda).$$

Now we take  $\Upsilon(\Lambda) = C_1(k, \|b\|_{L^2}, \Lambda)$  with  $\Lambda = C_0(k, \|b\|_{L^2})$ , then it is easy to see that  $u \in \mathcal{X}$  and  $\mathcal{X}$  is a nonempty bounded and closed convex set in  $C^1[0, 1]$ . On the other hand, by the Arzelà-Ascoli theorem, the imbedding  $C^{1+\alpha}[0, 1] \hookrightarrow C^1[0, 1]$  is compact. Thus, the operator  $\mathcal{T}$  is a compact map of  $\mathcal{X}$  into itself. By Schauder fixed point theorem (see Corollary 2.3.10 in [6]), there exists a fixed point  $u \in \mathcal{X}$  such that

$$\mathcal{T}(u) = u.$$

Therefore, equation (2.26) has a weak solution  $u_k \in C^1[0, 1]$ , and  $\rho_k(x) = k/u_k(x)$  is a desired weak supersonic solution of (2.24).  $\square$

*Proof of Theorem 2.2.* Multiplying (2.26) by  $(u_k - k)$  and using Young's inequality, we have

$$\begin{aligned} & (k-1) \int_0^1 \frac{u_k+1}{u_k} |(u_k)_x|^2 dx + \frac{4}{9} \int_0^1 \frac{u_k+1}{u_k} |[(u_k-k)^{3/2}]_x|^2 dx \\ &= \int_0^1 \left(b - \frac{k}{u_k}\right) (u_k - k) dx \\ &\leq \frac{1}{3} \int_0^1 (u_k - k)^3 dx + \frac{2}{3} \int_0^1 \left(b - \frac{k}{u_k}\right)^{3/2} dx \\ &\leq \frac{1}{3} \int_0^1 |[(u_k - k)^{3/2}]_x|^2 dx + \frac{2}{3} \int_0^1 b^{3/2}(x) dx. \end{aligned}$$

Thus, we have

$$\|(k-1)^{\frac{1}{2}}(u_k)_x\|_{L^2} + \|(u_k - k)^{\frac{3}{2}}\|_{H^1} \leq C \quad (2.30)$$

for a constant  $C$  independent of  $k$ , where we have used  $k > 1$  and  $u_k \geq k$ . This inequality together with the Sobolev imbedding theorem yields

$$\|u_k\|_{L^\infty} \leq k + C^{\frac{2}{3}}. \quad (2.31)$$

Hence

$$\rho_k(x) = \frac{k}{u_k(x)} \geq \frac{k}{\|u_k\|_{L^\infty}} \geq \frac{k}{k + C^{\frac{2}{3}}} \geq \frac{1}{1 + C^{\frac{2}{3}}} \triangleq \ell, \quad \forall x \in [0, 1]. \quad (2.32)$$

A direct calculation yields

$$(\rho_k)_x = -\frac{k(u_k)_x}{u_k^2} \text{ and } ((1 - \rho_k)^2)_x = \frac{4k(u_k - 1)^{\frac{1}{2}}((u_k - 1)^{\frac{3}{2}})_x}{3u_k^3}.$$

It then follows from (2.30) and (2.31) that

$$\begin{aligned} \|(1 - \rho_k)^2\|_{H^1} + \|(1 - \rho_k)^{3/2}\|_{H^1} &\leq C_1, \\ \|(k - 1)(\rho_k)_x\|_{L^2} &\leq C_1(k - 1)^{\frac{1}{2}}. \end{aligned}$$

Thus, there exists a function  $\rho_{sup}(x)$  such that, as  $k \rightarrow 1^+$ , up to a subsequence,

$$\begin{aligned} (1 - \rho_k)^2 &\rightharpoonup (1 - \rho_{sup})^2 \text{ weakly in } H^1(0, 1), \\ (1 - \rho_k)^{3/2} &\rightharpoonup (1 - \rho_{sup})^{3/2} \text{ weakly in } H^1(0, 1), \\ (1 - \rho_k)^{3/2} &\rightarrow (1 - \rho_{sup})^{3/2} \text{ strongly in } C^{\frac{1}{2}}[0, 1] \\ (k - 1)(\rho_k)_x &\rightarrow 0 \text{ strongly in } L^2(0, 1). \end{aligned} \tag{2.33}$$

Applying the same procedure as the proof of Theorem 2.1, one can show that  $\rho_{sup}$  satisfies (1.8).

The lower bound of  $\rho_{sup}$  follows from (2.32) and the third convergence of (2.33).

Let us now prove that  $\rho_{sup}(x) < 1$  for any interior point  $x \in (0, 1)$ . Observing that if a function  $\rho$  satisfies  $\rho(x) \equiv 1$  on an interval  $[\hat{a}, \hat{c}] \subset [0, 1]$ , then  $\rho$  is not a solution of equation (1.7) because  $\underline{b} > 1$ . Thus, for any  $1 \gg \epsilon > 0$ , there exists a  $\delta > 0$  and two points  $\hat{a}_\epsilon \in (0, \epsilon]$  and  $\hat{c}_\epsilon \in [1 - \epsilon, 1)$  such that  $\rho_{sup}(\hat{a}_\epsilon), \rho_{sup}(\hat{c}_\epsilon) \leq 1 - \delta < 1$ . We only need to show that  $\rho_{sup}(x) \leq 1 - \delta$  over  $[\hat{a}_\epsilon, \hat{c}_\epsilon]$ . Actually, set  $w := (1 - \rho_{sup})^2$ , then  $w \in H_0^1(0, 1)$ ,  $w(\hat{a}_\epsilon), w(\hat{c}_\epsilon) \geq \delta^2$  and it follows from (1.8) that for any  $\varphi \in H_0^1(\hat{a}_\epsilon, \hat{c}_\epsilon)$

$$\frac{1}{2} \int_{\hat{a}_\epsilon}^{\hat{c}_\epsilon} \frac{2 - \sqrt{w}}{(1 - \sqrt{w})^3} w_x \varphi_x dx + \frac{1}{\tau} \int_{\hat{a}_\epsilon}^{\hat{c}_\epsilon} \frac{\varphi_x}{1 - \sqrt{w}} dx + \int_{\hat{a}_\epsilon}^{\hat{c}_\epsilon} (1 - \sqrt{w} - b) \varphi dx = 0.$$

Taking  $\varphi(x) = (w - \delta^2)^-(x)$ , then

$$\frac{1}{2} \int_{\hat{a}_\epsilon}^{\hat{c}_\epsilon} \frac{2 - \sqrt{w}}{(1 - \sqrt{w})^3} |(w - \delta^2)^-|_x|^2 dx + \frac{1}{\tau} \int_{\hat{a}_\epsilon}^{\hat{c}_\epsilon} \frac{[(w - \delta^2)^-]_x}{1 - \sqrt{w}} dx + \int_{\hat{a}_\epsilon}^{\hat{c}_\epsilon} (1 - \sqrt{w} - b)(w - \delta^2)^- dx = 0.$$

Observing that  $\rho_{sup} \geq \ell$ , hence  $2 - \sqrt{w} > 1 - \sqrt{w} \geq \ell > 0$ . This implies that the first term of the equality is non-negative. Because  $b > \underline{b} > 1$ , the third term is also non-negative. On the other hand a simple computation gives  $-2(\sqrt{w} + \ln(1 - \sqrt{w}))_x = \frac{w_x}{1 - \sqrt{w}}$ , which implies the second term is zero. Thus,  $(w - \delta^2)^-(x) = 0$  over  $[\hat{a}_\epsilon, \hat{c}_\epsilon]$ . And as a result,  $\rho_{sup}(x) \leq 1 - \delta$  over  $[\hat{a}_\epsilon, \hat{c}_\epsilon]$ .

It is left to show (ii). We only need to show that if  $z_0 \in (0, 1)$  is a critical point of  $\rho_{sup}$ , then it must be a local minimal point. Because  $\rho_{sup} \in C[0, 1]$  and  $\rho_{sup} < 1$  over  $(0, 1)$ , by the interior

regularity theory of elliptic equation and the Sobolev imbedding, for any  $z_0 \in (0, 1)$ , there exists an interval  $z_0 \in I \subset (0, 1)$  such that  $z_0 \in I$ ,  $\rho_{sup} \in W^{2,p}(I)$  for any  $1 < p < \infty$  and  $\rho_{sup} \in C^1(\bar{I})$ . Now if  $z_0$  is a critical point, then  $(\rho_{sup})_x(z_0) = 0$ . Since  $\rho_{sup} \in C^1(\bar{I})$ , there exists a  $\delta > 0$  such that

$$|(\rho_{sup})_x(x)| < \frac{\tau(\underline{b} - 1)}{2} \text{ for any } x \in (z_0 - \delta, z_0 + \delta).$$

If  $x \in (z_0, z_0 + \delta)$ , we integrate (1.7) over  $(z_0, x)$  to derive

$$\begin{aligned} \left( \frac{1}{\rho_{sup}} - \frac{1}{\rho_{sup}^3} \right) (\rho_{sup})_x &= \int_{z_0}^x \left[ \rho_{sup} - b + \frac{(\rho_{sup})_x}{\tau \rho^2} \right] ds \\ &< \int_{z_0}^x \left( 1 - \underline{b} + \frac{|\rho_x|}{\tau \rho^2} \right) ds \\ &< \int_{z_0}^x \left( 1 - \underline{b} + \frac{\underline{b} - 1}{2} \right) ds \\ &= \frac{(1 - \underline{b})(x - z_0)}{2} \\ &< 0, \end{aligned}$$

where we have used  $(\rho_{sup})_x(z_0) = 0$  and  $\rho_{sup} < 1$ . Thus,

$$(\rho_{sup})_x(x) > 0 \text{ on } (z_0, z_0 + \delta).$$

Similarly, integrating (1.7) over  $(z_0 - \delta, x)$ , one can get that

$$(\rho_{sup})_x(x) < 0 \text{ on } (z_0 - \delta, z_0).$$

Therefore,  $z_0$  is a local minimal point of  $\rho_{sup}$ . The proof is complete.  $\square$

As in Proposition 2.1, we also study the optimal global regularity of the interior supersonic solution.

**Proposition 2.2.**  $\rho_{sup} \in C^{1/2}[0, 1]$ , and there exist  $s_2 \ll 1$ ,  $C_i$  ( $i = 5, 6, 7, 8$ ) such that

$$\begin{aligned} -C_5 x^{1/2} &< \rho_{sup} - 1 < -C_6 x^{1/2}, \\ -C_7 x^{-1/2} &< (\rho_{sup})_x < -C_8 x^{-1/2}, \end{aligned} \text{ for } x \in [0, s_2]. \quad (2.34)$$

*Proof.* The proof is similar to that of Proposition 2.1. Here for supersonic solutions, we need the local analysis for the solution near  $x = 0$ . We omit the details.  $\square$

### 2.3 Infinitely many transonic solutions with shocks

We turn to study the existence of transonic solutions of (1.5)-(1.6). We first consider Euler-Poisson equations (1.5) with constant doping profile  $\underline{b}$  but without the semiconductor effect



(namely  $\frac{1}{\tau} = 0$ , or say  $\tau = \infty$ ), and the boundary condition subjected is completely supersonic.

That is

$$\begin{cases} \left(1 - \frac{1}{\rho^2}\right) \rho_x = \rho E, \\ E_x = \rho - \underline{b}, \\ \rho(0) = \rho(L) = 1 - \delta, \end{cases} \quad (\text{supersonic boundary}), \quad (2.35)$$

where  $L \geq \frac{1}{4}$  is the parameter of length and  $\delta > 0$  is a small constant. As shown in the proof of Theorem 2.2, for any  $\delta > 0$ , (2.35) has a supersonic solution. We have the following uniform estimates with respect to  $\delta$  for the supersonic solutions of (2.35).

**Lemma 2.4.** *Assume that  $\underline{b} > 1$ , and that  $(\rho_L, E_L)(x)$  are supersonic solutions of (2.35). Then*

$$\beta(L, \underline{b}) \leq \min_{x \in [0, L]} \rho_L(x) \leq \gamma(L, \underline{b}), \quad \text{and } E_L(0) \geq C(L, \underline{b}),$$

where  $\beta(L, \underline{b})$ ,  $\gamma(L, \underline{b})$  and  $C(L, \underline{b})$  are positive constants independent of  $\delta$ .

*Proof.* For convenience, we denote  $(\rho_L, E_L)$  by  $(\rho, E)$ . In the phase-plane  $(\rho, \mathbf{E})$ , we have

$$\frac{d\mathbf{E}}{d\rho} = \frac{(\rho + 1)(\rho - \underline{b})(\rho - 1)}{\mathbf{E}\rho^3}.$$

Integrating the above equation with respect to  $\rho$ , we obtain the part of trajectory through  $(1 - \delta, E(0))$  as follows

$$\frac{E^2(x)}{2} = \frac{E^2(0)}{2} - \frac{2\rho(0) - \underline{b}}{2\rho^2(0)} - \rho(0) + \underline{b} \ln \rho(0) + \frac{2\rho(x) - \underline{b}}{2\rho^2(x)} + \rho(x) - \underline{b} \ln \rho(x), \quad (2.36)$$

and

$$E(x) = \pm 2 \sqrt{\frac{E^2(0)}{2} - \frac{2\rho(0) - \underline{b}}{2\rho^2(0)} - \rho(0) + \underline{b} \ln \rho(0) + \frac{2\rho(x) - \underline{b}}{2\rho^2(x)} + \rho(x) - \underline{b} \ln \rho(x)}.$$

Thus, all trajectories are symmetric with respect to  $\mathbf{E} \equiv 0$  and the supersonic solution obtained satisfies  $0 < \rho(x) < 1 - \delta$  and is symmetric in  $x \in (0, L)$ . Set  $\underline{\rho} := \min_{x \in [0, L]} \rho(x)$ , by the symmetry of  $\rho(x)$  in  $(0, L)$ , we know that  $\rho(x)$  reaches its minimum at  $x = \frac{L}{2}$ . Thus,

$$\underline{\rho} = \rho(L/2) \quad \text{and} \quad \rho'(L/2) = 0. \quad (2.37)$$

We next estimate  $\underline{\rho}$ . The velocity  $u(x) = 1/\rho(x)$  satisfies  $u(x) \geq \frac{1}{1-\delta}$  and

$$\left( \left( u - \frac{1}{u} \right) u_x \right)_x = \frac{1 - \underline{b}u}{\lambda u}, \quad u(0) = u(L) = \frac{1}{1-\delta}. \quad (2.38)$$

Multiplying (2.38) by  $(u - \frac{1}{1-\delta})^2$ , we get

$$2 \int_0^L \left( u - \frac{1}{u} \right) \left( u - \frac{1}{1-\delta} \right) (u_x)^2 dx = \int_0^L \frac{\underline{b}u - 1}{u} \left( u - \frac{1}{1-\delta} \right)^2 dx. \quad (2.39)$$

Artfully, we can reduce the left-hand-side of (2.39) to

$$\begin{aligned}
& 2 \int_0^L \left(u - \frac{1}{u}\right) \left(u - \frac{1}{1-\delta}\right) (u_x)^2 dx \\
&= 2 \int_0^L \frac{u+1}{u} (u-1) \left(u - \frac{1}{1-\delta}\right) (u_x)^2 dx \\
&= 2 \int_0^L \frac{u+1}{u} \left(\frac{\delta}{1-\delta} + u - \frac{1}{1-\delta}\right) \left(u - \frac{1}{1-\delta}\right) (u_x)^2 dx \\
&= \frac{2\delta}{1-\delta} \int_0^L \frac{u+1}{u} \left(u - \frac{1}{1-\delta}\right) (u_x)^2 dx \\
&\quad + 2 \int_0^L \frac{u+1}{u} \left(u - \frac{1}{1-\delta}\right)^2 (u_x)^2 dx \\
&= \frac{2\delta}{1-\delta} \int_0^L \frac{u+1}{u} \left(u - \frac{1}{1-\delta}\right) (u_x)^2 dx \\
&\quad + \frac{1}{2} \int_0^L \frac{u+1}{u} \left| \left(u - \frac{1}{1-\delta}\right)^2 \right|_x^2 dx,
\end{aligned} \tag{2.40}$$

and by using Cauchy-Schwarz's inequality and Poincaré's inequality, we can estimate the right-hand-side of (2.39) as follows

$$\begin{aligned}
& \int_0^L \frac{bu-1}{u} \left(u - \frac{1}{1-\delta}\right)^2 dx \\
&\leq \frac{1}{2L^2} \int_0^L \left(u - \frac{1}{1-\delta}\right)^4 dx + \frac{b^2 L^3}{2} \\
&\leq \frac{1}{4} \int_0^L \left| \left(u - \frac{1}{1-\delta}\right)^2 \right|_x^2 dx + \frac{b^2 L^3}{2}.
\end{aligned} \tag{2.41}$$

Substituting (2.40) and (2.41) to (2.39), we then have

$$\frac{2\delta}{1-\delta} \int_0^L \frac{u+1}{u} \left(u - \frac{1}{1-\delta}\right) u_x^2 dx + \int_0^L \frac{u+2}{4u} \left| \left(u - \frac{1}{1-\delta}\right)^2 \right|_x^2 dx \leq \frac{b^2 L^3}{2},$$

which gives

$$\left\| \left(u - \frac{1}{1-\delta}\right)^2 \right\|_{L^2(0,L)} \leq bL\sqrt{2L}. \tag{2.42}$$

Notice that, for  $\phi \in H_0^1(0, L)$ , Sobolev's inequality

$$\|\phi\|_{L^\infty} \leq \sqrt{2} \|\phi\|_{L^2}^{1/2} \|\phi_x\|_{L^2}^{1/2}$$

and Poincaré's inequality

$$\|\phi\|_{L^2} \leq 2L \|\phi_x\|_{L^2}$$

imply

$$\|\phi\|_{L^\infty} \leq 2\sqrt{L} \|\phi_x\|_{L^2}.$$

Thus, from (2.42) we have

$$\left(u(x) - \frac{1}{1-\delta}\right)^2 \leq 2\sqrt{L} \left\| \left(u - \frac{1}{1-\delta}\right)_x \right\|_{L^2(0,L)} \leq 2\sqrt{2}\underline{b}L^2,$$

which gives

$$u(x) \leq \frac{1}{1-\delta} + \sqrt{2\sqrt{2}\underline{b}} \cdot L.$$

Thus, we can estimate the minimum of  $\rho(x)$  by

$$\underline{\rho} \geq \left( \frac{1}{1-\delta} + \sqrt{2\sqrt{2}\underline{b}} \cdot L \right)^{-1} \geq \left( 2 + \sqrt{2\sqrt{2}\underline{b}} \cdot L \right)^{-1} \triangleq \beta(L), \text{ when } \delta \leq \frac{1}{2}.$$

On the other hand, by (2.35), since  $\underline{b} \geq 1 > \rho$ , we have

$$\rho_{xx} = \frac{\rho^3}{\rho+1} \left[ \frac{1}{\rho^2(1-\rho)} \left( \frac{3}{\rho^2} - 1 \right) \rho_x^2 + \frac{\underline{b}-\rho}{(1-\rho)} \right] \geq \frac{\rho^3}{2} \geq \frac{\beta^3(L)}{2} \text{ on } [0, L]. \quad (2.43)$$

By Taylor expansion

$$\rho(0) = \rho(L/2) - \rho'(L/2)L/2 + \rho''(\xi)(L/2)^2/2 \text{ with } \xi \in [0, L/2],$$

it then follows from (2.37) and (2.43) that

$$\underline{\rho} \leq 1 - \delta - \frac{L^2\beta^3(L)}{2^4} \leq 1 - \frac{L^2}{2^4} \cdot \frac{1}{(2 + \sqrt{2\sqrt{2}\underline{b}} \cdot L)^3} \triangleq \gamma(L). \quad (2.44)$$

Since  $\underline{\rho} = \rho(L/2)$  is the minimum value, from (2.35) and the fact  $\rho_x(L/2) = 0$ , we have  $E(L/2) = 0$ . Thus, in view of (2.36), we further obtain

$$\begin{aligned} \frac{E^2(0)}{2} &= \frac{2-\underline{b}-2\delta}{2(1-\delta)^2} + 1 - \delta - \underline{b}\ln(1-\delta) - \left[ \frac{2\underline{\rho}-\underline{b}}{2\underline{\rho}^2} + \underline{\rho} - \underline{b}\ln\underline{\rho} \right] \\ &= \frac{\delta[2-2\underline{b}-(2-\underline{b})\delta]}{2(1-\delta)^2} - \delta - \underline{b}\ln(1-\delta) + 1 \\ &\quad + \frac{(2-\underline{b})(\underline{\rho}-1)^2 + (2-2\underline{b})(\underline{\rho}-1)}{2\underline{\rho}^2} - \underline{\rho} + \underline{b}\ln\underline{\rho} \\ &\geq \frac{\delta[2-2\underline{b}-(2-\underline{b})\delta]}{2(1-\delta)^2} - \delta + f(\underline{\rho}), \end{aligned}$$

where

$$f(s) := 1 + \frac{(2-\underline{b})(s-1)^2 + (2-2\underline{b})(s-1)}{2s^2} - s + \underline{b}\ln s, \quad s \in (0, 1).$$

Notice that  $f(1) = 0$  and  $f'(s) := -\frac{(\underline{b}-s)(1-s^2)}{s^3} < 0$  for  $s \in (0, 1)$ , namely,  $f(s)$  is decreasing and positive for  $s \in (0, 1)$ , using the boundness estimates carried out in (2.44):  $\underline{\rho} \leq \gamma(L)$ , we have, when  $\delta$  is small such that  $\frac{\delta^2}{2(1-\delta)^2} + \delta \leq \frac{f(\gamma(L))}{2}$ , then

$$E^2(0) \geq 2 \left[ \frac{-\delta^2}{2(1-\delta)^2} - \delta + f(\underline{\rho}) \right] \geq 2 \left[ \frac{-\delta^2}{2(1-\delta)^2} - \delta + f(\gamma(L)) \right] \geq f(\gamma(L)). \quad (2.45)$$

Integrating the second equation of (2.35) over  $[0, L/2]$ , we get

$$E(0) = E(L/2) + \int_0^{L/2} (1 - \rho) dt = \int_0^{L/2} (1 - \rho) dt > 0.$$

Hence, it follows from (2.45) that,  $E(0)$  has a positive lower bound

$$E(0) \geq \sqrt{f(\gamma(L))}, \quad (2.46)$$

which is independent of  $\delta$ .  $\square$

**Theorem 2.3.** *If  $\underline{b} > 1$ ,  $\tau \gg 1$  and  $0 \leq \bar{b} - \underline{b} \ll 1$ , then system (1.5)-(1.6) has infinitely many transonic shock solutions over  $[0, 1]$ .*

*Proof.* The proof is technical and longer, we divide it into seven steps.

*Step 1.* Let  $\eta$  be a small number to be determined later such that  $\delta < \eta \ll 1$ . Denote by  $(\rho_1, E_1)(x)$  the solution of (2.35) with  $L = \frac{1}{2}$ . Then by (2.46),

$$E_1(0) \geq \sqrt{f(\gamma(1/2))} \triangleq \Lambda_1. \quad (2.47)$$

Let us consider system (1.5) with the supersonic initial value:

$$\begin{cases} \left(1 - \frac{1}{\rho^2}\right) \rho_x = \rho E - \frac{1}{\tau}, \\ E_x = \rho - b(x), \\ (\rho(0), E(0)) = (1 - \delta, E_1(0)). \end{cases} \quad (2.48)$$

In this step, we will show that when  $\tau \gg 1$ , there exists a number  $x_1 \leq C\eta$  such that  $\rho(x_1) = 1 - \eta$ , and  $E(x_1) \geq E_1(0) - C\eta^2$ , where  $C > 0$  is a constant independent of  $\tau$ ,  $\delta$  and  $\eta$ .

It is easy to see that if  $\tau \geq \frac{4}{\Lambda_1} \geq \frac{4}{E_1(0)}$  and  $\delta \leq \frac{1}{4}$ , then the initial data of (2.48) satisfies

$$\rho(0)E(0) - \frac{1}{\tau} = (1 - \delta)E_1(0) - \frac{1}{\tau} \geq \frac{E_1(0)}{2} > 0.$$

Observing that (2.48) is a standard initial value problem for ODE system without degeneracy, it follows that (2.48) has a unique supersonic solution on some interval. Because  $b \geq \underline{b} > 1 > \rho$ , the solution component  $E$  keeps decreasing. Using the result (ii) of Theorem 2.2,  $\rho$  is decreasing until it attains the unique critical point, after that  $\rho$  keeps increasing. Denote by  $x_1$  the first number that  $\rho(x)$  attains  $1 - \eta$ , namely  $\rho(x_1) = 1 - \eta$ . By the second equation of (2.48),

$$E(x) = E_1(0) + \int_0^x (\rho - b) ds \geq E_1(0) - \bar{b}x \text{ for } x \in (0, x_1).$$

Since  $\rho \in (1 - \eta, 1 - \delta)$  on  $(0, x_1)$ , if  $\eta \leq \frac{1}{2}$  and  $\tau \geq \frac{4}{\Lambda_1} \geq \frac{4}{E_1(0)}$ , then

$$\rho E - \frac{1}{\tau} \geq (1 - \eta)(E_1(0) - \bar{b}x) - \frac{E_1(0)}{4} \geq (1 - \eta) \left( \frac{E_1(0)}{4} - \bar{b}x \right) \text{ for } x \in (0, x_1),$$

which in combination with the first equation of (2.48) leads to

$$x_1 = \frac{\rho(x_1) - \rho(0)}{\rho_x(\xi)} = \frac{(\eta - \delta)(1 - \rho^2(\xi))}{(\rho(\xi)E(\xi) - \frac{1}{\tau})\rho^2(\xi)} \leq \frac{2\eta^2}{(1 - \eta)^3(\frac{E_1(0)}{4} - \bar{b}x_1)} \quad \text{with } \xi \in (0, x_1). \quad (2.49)$$

To solve this inequality, we notice that when  $\eta$  is small such that  $\eta \leq \min\{\frac{E_1(0)}{2^4\sqrt{b}}, \frac{1}{2}\}$ , then

$$\frac{E_1^2(0)}{4} - \frac{8\bar{b}\eta^2}{(1 - \eta)^3} \geq \frac{E_1^2(0)}{4} - 2^6\bar{b}\eta^2 \geq \frac{1}{4}(E_1^2(0) - 2^8\bar{b}\eta^2) \geq 0.$$

Thus, we get from inequality (2.49) that

$$\begin{aligned} x_1 &\leq \frac{1}{2\bar{b}} \left( \frac{E_1(0)}{4} - \left( \frac{E_1^2(0)}{4^2} - \frac{8\bar{b}\eta^2}{(1 - \eta)^3} \right)^{1/2} \right) \\ &= \frac{4\eta^2}{(1 - \eta)^3 \left( \frac{E_1(0)}{4} + \left( \frac{E_1^2(0)}{4^2} - \frac{8\bar{b}\eta^2}{(1 - \eta)^3} \right)^{1/2} \right)} \\ &\leq \frac{16\eta^2}{(1 - \eta)^3 E_1(0)} \leq \frac{2^7\eta^2}{\Lambda_1}, \end{aligned}$$

where we have used (2.47) in the last inequality. In view of the second equation of (2.48), we further get

$$E(x_1) = E_1(0) + \int_0^{x_1} (\rho - b)ds \geq E_1(0) - \bar{b}x_1 \geq E_1(0) - \frac{2^7\bar{b}\eta^2}{\Lambda_1}. \quad (2.50)$$

*Step 2.* Now let us consider the initial value problem for the ODE system without semiconductor effect

$$\begin{cases} \left(1 - \frac{1}{\hat{\rho}^2}\right) \hat{\rho}_x = \hat{\rho} \hat{E}, \\ \hat{E}_x = \hat{\rho} - \underline{b}, \\ (\hat{\rho}(0), \hat{E}(0)) = (1 - \delta, \hat{E}_0). \end{cases} \quad (2.51)$$

In this step, we prove that there exist numbers  $x_2 > 0$  and  $\hat{E}_0 > 0$  such that  $x_2 \leq C\eta^2$  and the solution of (2.51) satisfies  $\hat{\rho}(x_2) = 1 - \eta$  and  $\hat{E}(x_2) = E(x_1)$ . Here  $E$  and  $x_1$  are given by step 1, and  $C > 0$  is a constant independent of  $\tau$ ,  $\delta$  and  $\eta$ .

We argue by shooting method. Using phase-plane analysis, it is easy to see that, for any  $\hat{E}_0 > 0$ , there exists  $\hat{L}(\hat{E}_0) > 0$ , such that (2.51) has a symmetric supersonic solution on  $[0, \hat{L}(\hat{E}_0)]$  satisfying

$$\hat{\rho}(0) = \hat{\rho}(\hat{L}(\hat{E}_0)) = 1 - \delta, \quad \hat{E}(0) = \hat{E}(\hat{L}(\hat{E}_0)) = \hat{E}_0.$$

Now taking  $\hat{E}_0 = 2E_1(0)$ , suppose  $\bar{x}_2$  is the first number that  $\hat{\rho}$  attains  $1 - \eta$ , since  $\hat{\rho} \in (1 - \eta, 1 - \delta)$  on  $(0, \bar{x}_2)$ , by the second equation of (2.51),

$$\hat{E}(x) = \hat{E}(0) + \int_0^x (\hat{\rho} - b)ds \geq 2E_1(0) - \bar{b}x \quad \text{for } x \in (0, \bar{x}_2). \quad (2.52)$$

Hence

$$\hat{\rho}\hat{E}(x) \geq (1 - \eta)(2E_1(0) - \bar{b}x),$$

which in combination with the first equation of (2.51) leads to

$$\bar{x}_2 = \frac{\hat{\rho}(\bar{x}_2) - \hat{\rho}(0)}{\hat{\rho}_x(\hat{\xi})} = \frac{(\eta - \delta)(1 - \hat{\rho}^2(\hat{\xi}))}{\hat{\rho}^3(\hat{\xi})\hat{E}(\hat{\xi})} \leq \frac{2\eta^2}{(1 - \eta)^3(2E_1(0) - \bar{b}\bar{x}_2)}. \quad (2.53)$$

Notice that when  $\eta \leq \min\{\frac{E_1(0)}{4\sqrt{b}}, \frac{1}{2}\}$ , it holds that

$$4E_1^2(0) - \frac{8\bar{b}\eta^2}{(1 - \eta)^3} \geq 4(E_1^2(0) - 2^4\bar{b}\eta^2) \geq 0.$$

It then follows from (2.53) that

$$\begin{aligned} \bar{x}_2 &\leq \frac{1}{2\bar{b}} \left( 2E_1(0) - \left( 4E_1^2(0) - \frac{8\bar{b}\eta^2}{(1 - \eta)^3} \right)^{1/2} \right) \\ &= \frac{2\eta^2}{(1 - \eta)^3 \left( E_1(0) + (E_1^2(0) - \frac{2\bar{b}\eta^2}{(1 - \eta)^3})^{1/2} \right)} \\ &\leq \frac{2\eta^2}{(1 - \eta)^3 E_1(0)} \leq \frac{2^4\eta^2}{\Lambda_1}, \end{aligned} \quad (2.54)$$

where we have used (2.47) in the last inequality. This inequality together with (2.52) gives

$$\hat{E}(\bar{x}_2) \geq 2E_1(0) - \bar{b}\bar{x}_2 \geq 2E_1(0) - \frac{2^4\bar{b}\eta^2}{E_1(0)} \geq 2E_1(0) - \frac{2^4\bar{b}\eta^2}{E_1(0)} \cdot \frac{E_1^2(0)}{16\bar{b}} = E_1(0) > E(x_1).$$

Here we have used  $E_1(0) = E(0) > E(x_1)$  because  $E$  is decreasing.

On the other hand, if  $\hat{E}_0 = \frac{E_1(0)}{2}$ , by (2.50), one can easily see that if  $\eta < \frac{\Lambda_1}{2^4\bar{b}}$ , it holds  $E(x_1) > \frac{E_1(0)}{2}$ . Thus,  $\hat{E}(x) < \hat{E}_0 = \frac{E_1(0)}{2} < E(x_1)$  for any  $x > 0$  because  $\hat{E}$  is decreasing. Now by the continuity of the solution with respect to the initial data, there exist  $\hat{E}_0 \in (\frac{E_1(0)}{2}, 2E_1(0))$  and length  $\hat{L} > 0$  such that (2.51) has a supersonic solution  $(\hat{\rho}, \hat{E})$  satisfying

$$\hat{\rho}(0) = \hat{\rho}(\hat{L}) = 1 - \delta, \quad \hat{E}(0) = \hat{E}(\hat{L}) = \hat{E}_0.$$

Moreover, as in (2.54), there exists a number  $x_2 \leq C\eta^2$  such that

$$\hat{\rho}(x_2) = 1 - \eta \quad \text{and} \quad \hat{E}(x_2) = E(x_1). \quad (2.55)$$

Thus,

$$0 < \hat{E}_0 - E(x_1) = \hat{E}_0 - \hat{E}(x_2) = -\hat{E}_x x_2 = (b - \hat{\rho})x_2 < \bar{b}x_2 < C_1\eta^2,$$

which in combination with (2.50) yields

$$\begin{aligned} \hat{E}_0 - E_1(0) &= \hat{E}_0 - E(x_1) + E(x_1) - E_1(0) > E(x_1) - E_1(0) > -C_2\eta^2, \\ \hat{E}_0 - E_1(0) &= \hat{E}_0 - E(x_1) + E(x_1) - E_1(0) < C_1\eta^2, \end{aligned}$$

where we have used the fact that  $E$  is decreasing. Thus

$$|\hat{E}_0 - E_1(0)| \leq C_3 \eta^2 \text{ with } C_3 = \min\{C_1, C_2\}.$$

Observing that the length  $\hat{L}$  of solution is also continuous with respect to the initial data, since the length of the solution  $(\rho_1, E_1)$  to (2.35) with initial data  $(1 - \delta, E_1(0))$  is  $\frac{1}{2}$ , there exists  $l_0 > 0$  independent of  $\tau, \delta$  and  $\eta$ , such that if  $C_3 \eta^2 < l_0$ , then

$$\frac{1}{4} \leq \hat{L} \leq \frac{3}{4}.$$

*Step 3.* In this step, we show that when  $\tau \gg 1$  and  $\bar{b} - \underline{b} \ll 1$ , system (2.48) has a unique solution  $(\rho, E)$  on  $[0, x_4]$  with

$$\frac{1}{4} - C\eta^2 \leq x_4 \leq \frac{3}{4} + C\eta^2, \quad \rho(0) = \rho(x_4) = 1 - \delta,$$

for some constant  $C$  independent of  $\tau, \delta$  and  $\eta$ . Set  $(\bar{\rho}, \bar{E})(x) := (\hat{\rho}, \hat{E})(x - x_1 + x_2)$ , then  $(\bar{\rho}, \bar{E})$  satisfies (2.51) with initial-boundary data

$$(\bar{\rho}, \bar{E})(x_1) = (1 - \eta, \hat{E}(x_2)) = (\rho, E)(x_1) \quad \text{and} \quad \bar{\rho}(x_3) = 1 - \eta$$

with  $x_3 := \hat{L} + x_1 - 2x_2$ , where we have used the symmetry of  $(\hat{\rho}, \hat{E})$ , and hence  $\hat{\rho}(\hat{L} - x_2) = \hat{\rho}(x_2) = 1 - \eta$ . Set  $\phi := \bar{\rho} - \rho, \psi := \bar{E} - E$ , then by (2.48) and (2.51),  $(\phi, \psi)$  satisfies

$$\begin{cases} \phi_x = \frac{\bar{\rho}^3 \psi}{(\bar{\rho}+1)(\bar{\rho}-1)} + \frac{(\bar{\rho}^2 \rho^2 - \bar{\rho}^2 - \bar{\rho} \rho - \rho^2) \phi E}{(\bar{\rho}+1)(\bar{\rho}-1)(\rho+1)(\rho-1)} + \frac{\rho^2}{\tau(\rho+1)(\rho-1)}, \\ \psi_x = \phi + \bar{b} - \underline{b}, \\ (\phi(x_1), \psi(x_1)) = 0. \end{cases} \quad (2.56)$$

Define the solution space  $X_T := \{(\phi, \psi) \in C[x_1, T] | \phi(x_1) = \psi(x_1) = 0, |\phi| \leq \eta/2, |\psi| \leq \eta/2\}$ . We only need to show the a priori estimate

$$\phi^2(x) + \psi^2(x) \leq \eta^2/4 \quad \text{on } x \in [x_1, x_3]. \quad (2.57)$$

Multiplying the first equation of (2.56) by  $\phi$  and the second one by  $\psi$  and adding them, noting  $|\rho - \bar{\rho}| \leq \eta/2$ , by Young's inequality, one can easily get

$$(\phi^2 + \psi^2)_x \leq \frac{C}{\eta^2} (\phi^2 + \psi^2) + \frac{C}{\tau^2} + C(\bar{b} - \underline{b})^2,$$

where  $C$  is a constant independent of  $\tau, \delta$  and  $\eta$ . It then follows from Gronwall's inequality that

$$\phi^2 + \psi^2 \leq C \left[ \frac{C}{\tau^2} + (\bar{b} - \underline{b})^2 \right] \eta^2 e^{Cx/\eta^2} \leq C \left[ \frac{C}{\tau^2} + (\bar{b} - \underline{b})^2 \right] \eta^2 e^{C/\eta^2} \text{ for } x \in [x_1, x_3].$$

Now taking  $\tau \gg 1$  and  $\bar{b} - \underline{b} \ll 1$  such that  $\left[ \frac{C}{\tau^2} + (\bar{b} - \underline{b})^2 \right] e^{C/\eta^2} \leq \frac{1}{4}$ , we derive (2.57). Moreover, we also get

$$|\rho - \bar{\rho}| \leq \eta/2 \text{ and } |E - \bar{E}| \leq \eta/2,$$

which gives  $\rho(x_3) \leq \bar{\rho}(x_3) + \frac{\eta}{2} = 1 - \frac{\eta}{2}$ , and further by (2.55) and (2.50),

$$E(x_3) \leq \bar{E}(x_3) + \frac{\eta}{2} = \hat{E}(\hat{L} - x_2) + \frac{\eta}{2} = -\hat{E}(x_2) + \frac{\eta}{2} = -E(x_1) + \frac{\eta}{2} \leq -E_1(0) + C\eta. \quad (2.58)$$

Now taking  $x_3$  as the initial data, we can extend  $(\rho, E)$ , the solution of (2.48), to the state satisfying  $\rho = 1 - \delta$ . Denote by  $x_4$  the number that  $\rho(x_4) = 1 - \delta$ . As in the proof of step 2, we have

$$x_4 - x_3 \leq C\eta^2$$

for some constant  $C$  independent of  $\tau$ ,  $\delta$  and  $\eta$ . And then by (2.58),

$$E(x_4) \leq E(x_3) \leq -E_1(0) + C\eta.$$

Now we obtain a solution of (2.48) on  $[0, x_4]$  satisfying

$$\rho(0) = \rho(x_4) = 1 - \delta, \quad E(0) = E_1(0), \quad E(x_4) \leq -E_1(0) + C\eta. \quad (2.59)$$

Moreover,

$$\frac{1}{4} - C\eta^2 \leq \hat{L} + x_1 - 2x_2 = x_3 \leq x_4 \leq x_3 + C\eta^2 = \hat{L} + x_1 - 2x_2 + C\eta^2 \leq \frac{3}{4} + C\eta^2. \quad (2.60)$$

*Step 4.* In this step, we construct a transonic solution of (2.48) on an interval  $[0, x_5]$  with

$$\frac{1}{4} - C\eta \leq x_5 \leq \frac{3}{4} + C\eta, \quad \rho(0) = 1 - \delta, \quad \rho(x_5) = 1 + \delta.$$

Set  $\rho_l = 1 - \eta$ , then  $\rho_r = 1/\rho_l > 1$ . We take the jump location  $\bar{x}_0 \in (0, x_4)$  as the last number that  $\rho(\bar{x}_0) = \rho_l$ , and restrict our supersonic solution  $(\rho_{sup}, E_{sup})(x)$  only on  $[0, \bar{x}_0]$ . We denote  $E_{sup}(\bar{x}_0) \triangleq E_l$ . As in the proof of step 2,

$$x_4 - \bar{x}_0 \leq C\eta. \quad (2.61)$$

Thus, owing to the inequality of (2.59), the supersonic solution satisfies

$$\rho_l = 1 - \eta, \quad E_l \leq E(x_4) + C\eta \leq -E_1(0) + C\eta. \quad (2.62)$$

It is then easy to see that

$$\rho_l E_l - \frac{1}{\tau} \leq (1 - \eta)(-E_1(0) + C\eta) = -E_1(0) + (C + E_1(0))\eta.$$

Thus, when  $\eta \ll 1$  such that  $(C + E_1(0))\eta \leq \frac{E_1(0)}{2}$ , it holds

$$\rho_l E_l - \frac{1}{\tau} \leq -\frac{E_1(0)}{2} < 0 \quad \text{and} \quad E_l < 0. \quad (2.63)$$



Next we construct the corresponding subsonic solution. For  $x \geq \bar{x}_0$ , let us consider the system (2.48) with the initial data

$$\rho(\bar{x}_0) = \rho_r, \quad E(\bar{x}_0) = E_r = E_l.$$

By the standard ODE theory, the initial value problem admits a unique solution  $(\rho, E)(x)$  for  $x > \bar{x}_0$ . By (2.63), a simple calculation gives

$$\begin{aligned} \rho_r E_r - \frac{1}{\tau} &= \rho_l E_l - \frac{1}{\tau} + \left(\frac{1}{\rho_l} - \rho_l\right) E_l \\ &\leq -\frac{E_1(0)}{2} + \left[\frac{1}{1-\eta} - (1-\eta)\right](-E_1(0) + C\eta) \\ &\leq -\frac{E_1(0)}{2} + C\eta^2. \end{aligned}$$

It hence follows that when  $C\eta^2 < \frac{E_1(0)}{4}$ ,

$$\rho_r E_r - \frac{1}{\tau} \leq -\frac{E_1(0)}{4} < 0.$$

From the first equation of (2.48), we know the component  $\rho$  of such solution is decreasing in a neighborhood of  $\bar{x}_0^+$ . We denote this subsonic solution by  $(\rho_{sub}, E_{sub})(x)$ . If  $\eta < 1 - \frac{1}{b}$ , then

$$\begin{aligned} E_{sub}(x) &= E_r + \int_{\bar{x}_0}^x (\rho_{sub} - b) dx \\ &\leq E_r + \int_{\bar{x}_0}^x \left(\frac{1}{1-\eta} - b\right) dx \\ &< E_r < 0, \end{aligned}$$

where we have used the second inequality of (2.63) and  $E_r = E_l$ . Noting that the function  $g(s) = \frac{s^3}{s^2-1}$  is monotone decreasing on  $(1, \sqrt{3})$ , we thus get from (2.62) that

$$(\rho_{sub})_x = \frac{\rho_{sub} E_{sub} - \frac{1}{\tau}}{1 - \frac{1}{\rho_{sub}^2}} \leq \frac{\rho_r^3 E_r}{\rho_r^2 - 1} = \frac{E_r}{\eta(1-\eta)(2-\eta)} \leq \frac{-E_1(0) + C\eta}{\eta(1-\eta)(2-\eta)} < -\frac{E_1(0)}{2},$$

if  $\eta < \min\left\{\frac{E_1(0)}{2C}, \frac{1}{2}\right\}$ . This inequality implies  $\rho_{sub}$  will keep decreasing and attains  $1 + \delta$  at a finite number  $x_5$  and

$$x_5 - \bar{x}_0 = \frac{\delta - \frac{\eta}{1-\eta}}{\int_0^1 (\rho_{sub})_x (sx_5 + (1-s)\bar{x}_0) ds} \leq C\eta^2 \text{ if } \eta < \min\left\{\frac{E_1(0)}{2C}, \frac{1}{2}\right\}. \quad (2.64)$$

Now we have constructed the transonic solution to (2.48) in  $[0, x_5]$  as follows

$$(\rho_{trans}, E_{trans})(x) = \begin{cases} (\rho_{sup}, E_{sup})(x), & x \in [0, \bar{x}_0), \\ (\rho_{sub}, E_{sub})(x), & x \in (\bar{x}_0, x_5], \end{cases}$$

which satisfies the boundary condition

$$\rho_{sup}(0) = 1 - \delta, \quad \rho_{sub}(x_5) = 1 + \delta,$$

and the entropy condition at  $\bar{x}_0$

$$0 < \rho_{sup}(\bar{x}_0^-) = 1 - \eta < 1 < \rho_{sub}(\bar{x}_0^+),$$

and the Rankine-Hugoniot condition (1.11) at  $\bar{x}_0$ . Furthermore, it follows from (2.60), (2.61) and (2.64) that

$$\frac{1}{4} - C\eta \leq x_5 \leq \frac{3}{4} + C\eta.$$

*Step 5.* In this step, we construct a transonic solution of (2.48) on an interval  $[0, x_7]$  with  $\frac{5}{4} - C\eta \leq x_7 \leq \frac{7}{4} + C\eta$ ,  $\rho(0) = 1 - \delta$  and  $\rho(x_7) = 1 + \delta$ .

We take  $L = \frac{3}{2}$  in (2.35) and denote by  $(\rho_2, E_2)$  its solution. As shown in steps 1-3, we know that there exists an interval  $[0, x_6]$  with

$$\frac{5}{4} - C\eta^2 \leq x_6 \leq \frac{7}{4} + C\eta^2,$$

such that system (2.48) has a supersonic solution on  $[0, x_6]$  satisfying

$$\rho(0) = \rho(x_6) = 1 - \delta, \quad E(0) = E_2(0), \quad E(x_6) \leq -E_2(0) + C\eta.$$

As in step 4, we may construct another transonic solution for (2.48) in the form of

$$(\rho_{trans}, E_{trans})(x) = \begin{cases} (\rho_{sup}, E_{sup})(x), & x \in [0, \tilde{x}_0), \\ (\rho_{sub}, E_{sub})(x), & x \in (\tilde{x}_0, x_7], \end{cases}$$

where  $\tilde{x}_0 \in (0, x_6)$  and  $\frac{5}{4} - C\eta^2 \leq x_7 \leq \frac{7}{4} + C\eta^2$  are some determined numbers. This transonic solution satisfies the boundary condition

$$\rho_{sup}(0) = 1 - \delta, \quad \rho_{sub}(x_7) = 1 + \delta,$$

the entropy condition at  $\tilde{x}_0$

$$0 < \rho_{sup}(\tilde{x}_0^-) = 1 - \eta < 1 < \rho_{sub}(\tilde{x}_0^+),$$

and the Rankine-Hugoniot condition (1.11) at  $\tilde{x}_0$ .

*Step 6.* We next construct transonic solutions of (2.48) on  $[0, 1]$ . Without loss of generality, we assume that  $E_1(0) < E_2(0)$ . As in step 4, one can see that when  $0 < \delta < \eta \ll 1$ ,  $\tau \gg 1$ , for any  $E_0 \in (E_1(0), E_2(0))$ , there exists a number  $x_8 > 0$  and a transonic solution of (2.48) on the interval  $[0, x_8]$  satisfying the boundary condition

$$\rho_{sup}(0) = 1 - \delta, \quad E_{sup}(0) = E_0, \quad \rho_{sub}(x_8) = 1 + \delta,$$

the entropy condition at  $\tilde{x}_0$

$$0 < \rho_{sup}(\tilde{x}_0^-) = 1 - \eta < 1 < \rho_{sub}(\tilde{x}_0^+),$$

and the Rankine-Hugoniot condition. Applying the continuation argument in the length of the interval  $L$ , we realize that (2.48) has some transonic solutions  $(\rho_{trans}, E_{trans})(x)$  for  $x \in [0, 1]$  and satisfies the boundary condition

$$\rho_{sup}(0) = 1 - \delta, \quad \rho_{sub}(1) = 1 + \delta,$$

the entropy condition

$$0 < \rho_{sup}(x_0^\delta) = 1 - \eta < 1 < \rho_{sub}(x_0^\delta),$$

and the Rankine-Hugoniot condition at some jump location  $x_0^\delta$  in  $(0, 1)$ .

*Step 7.* Let us now prove the existence of transonic solutions of (1.5)-(1.6) on  $[0, 1]$ .

For any  $\delta > 0$ , denote by  $(\rho^\delta, E^\delta)$  the transonic solution of (2.48) on  $[0, 1]$  obtained in step 6. Multiplying the first equation of (2.48) by  $((1 - \delta - \rho^\delta)^2)_x$ , integrating the resultant equation on  $(0, x_0^\delta)$ , and using the second equation of (2.48), noting

$$\begin{aligned} \frac{((1 - \delta - \rho^\delta)^2)_x}{\rho^\delta} &= (-2(1 - \delta) \ln \rho^\delta + 2\rho^\delta)_x, \\ \int_0^{x_0^\delta} (b - \rho^\delta)(1 - \delta - \rho^\delta)^2 dx &\leq \int_0^{x_0^\delta} b(1 - \delta - \rho^\delta)^2 dx \\ &\leq \frac{1}{4} \int_0^{x_0^\delta} (1 - \delta - \rho^\delta)^4 dx + \int_0^{x_0^\delta} b^2 dx \\ &\leq \frac{1}{4} \int_0^{x_0^\delta} |((1 - \delta - \rho^\delta)^2)_x|^2 dx + b^2, \end{aligned}$$

we have

$$\begin{aligned} &\int_0^{x_0^\delta} \frac{2\delta(\rho^\delta + 1)(1 - \delta - \rho^\delta)(\rho_x)^2}{(\rho^\delta)^3} + \frac{(\rho^\delta + 1)|((1 - \delta - \rho^\delta)^2)_x|^2}{2(\rho^\delta)^3} dx \\ &\leq \frac{1}{4} \int_0^{x_0^\delta} |((1 - \delta - \rho^\delta)^2)_x|^2 dx + b^2 + E_l(1 - \delta - \rho_l)^2 \\ &\quad - \frac{2}{\tau} [-(1 - \delta) \ln \rho_l + \rho_l + (1 - \delta) \ln(1 - \delta) - (1 - \delta)]. \end{aligned} \tag{2.65}$$

Similarly, multiplying the first equation of (2.48) by  $((\rho^\delta - 1 - \delta)^2)_x$ , integrating the resultant equation on  $(x_0^\delta, 1)$ , we have

$$\begin{aligned} &\int_{x_0^\delta}^1 \frac{2\delta(\rho^\delta + 1)(\rho^\delta - 1 - \delta)((\rho^\delta)_x)^2}{\rho^3} + \frac{(\rho^\delta + 1)|((\rho^\delta - 1 - \delta)^2)_x|^2}{2(\rho^\delta)^3} dx \\ &\leq \frac{1}{4} \int_{x_0^\delta}^1 |((\rho^\delta - 1 - \delta)^2)_x|^2 dx + b^2 - E_r(\rho^\delta - 1 - \delta)^2 \\ &\quad + \frac{2}{\tau} [\rho_r - (1 + \delta) \ln \rho_r - 1 + (1 + \delta) \ln(1 + \delta)]. \end{aligned}$$

Substituting this inequality into (2.65), we get

$$\|(1 - \delta - \rho_{sup}^\delta)^2\|_{H^1(0, x_0^\delta)} \leq C, \quad \|(\rho_{sub}^\delta - 1 - \delta)^2\|_{H^1(x_0^\delta, 1)} \leq C.$$

Since  $\eta > 0$ , as  $\delta \rightarrow 0^+$ , up to a subsequence,  $x_0^\delta \rightarrow x_0 \in (0, 1)$ . Thus, for integer  $k$  large enough, there exists a subsequence, still denoted by  $\{\rho^\delta\}$  such that

$$\begin{aligned} (1 - \delta - \rho_{sup}^\delta)^2 &\rightharpoonup (1 - \rho_{sup}^0)^2 \quad \text{weakly in } H^1(0, x_0 - 1/k), \\ (\rho_{sub}^\delta - 1 - \delta)^2 &\rightharpoonup (\rho_{sub}^0 - 1)^2 \quad \text{weakly in } H^1(x_0 + 1/k, 1). \end{aligned}$$

Applying the diagonal argument for  $(\rho_{trans}^\delta, E_{trans}^\delta)$ , we know that (1.5)-(1.6) has a transonic solution  $(\rho_{trans}, E_{trans})(x)$  for  $x \in [0, 1]$  that satisfies the sonic boundary condition, the entropy condition and the Rankine-Hugoniot condition at the jump location  $x_0$  in  $(0, 1)$ .

Because  $\tau$  only depends on  $(E_1(0), E_2(0), \eta)$ , and  $\eta$  only depends on  $(E_1(0), E_2(0))$ , there exists a  $\eta_0 > 0$  such that for any  $\eta \in (0, \eta_0)$ , there exists a transonic solution jumps at  $\rho_l = 1 - \eta$ . Thus, such transonic solutions are infinitely many due to arbitrary choice of  $0 < \eta < \eta_0$ . The proof is complete.  $\square$

## 2.4 Infinitely many $C^1$ transonic solutions

In this subsection, we assume that the doping profile  $b(x) = b > 1$  is a given constant. We will construct  $C^1$  smooth transonic solution on the base of refined local analysis of the interior subsonic solutions and interior supersonic solutions on the boundary. The approach highly relies on the phase-plane analysis.

We first study the structure of interior subsonic solution. For convenience, we set

$$F = E - \frac{1}{\tau\rho} \quad \text{and} \quad n = \rho - 1. \quad (2.66)$$

Then system (1.5) is transformed to

$$\begin{cases} n_x = \frac{(1+n)^3 F}{(2+n)n}, \\ F_x = n + 1 - b + \frac{(1+n)F}{\tau(2+n)n}. \end{cases} \quad (2.67)$$

Clearly,  $(b-1, 0)$  is a saddle point of (2.67). In the  $(n, \mathbf{F})$  plane, all trajectories satisfy

$$\begin{aligned} \frac{d\mathbf{F}}{dn} &= \frac{(n+1-b)(2+n)}{(1+n)^3} \cdot \frac{n}{\mathbf{F}} + \frac{1}{\tau(1+n)^2} \\ &\triangleq H_1(n, \mathbf{F}). \end{aligned} \quad (2.68)$$

Here and in the sequel, to avoid confusion,  $\mathbf{F} = \mathbf{F}(n)$  denotes the function of the trajectory. The equation  $H_1(n, \mathbf{F}) = 0$  determines a curve

$$\Xi = \Xi(n) = -\frac{\tau(n+1-b)(2+n)n}{1+n}. \quad (2.69)$$

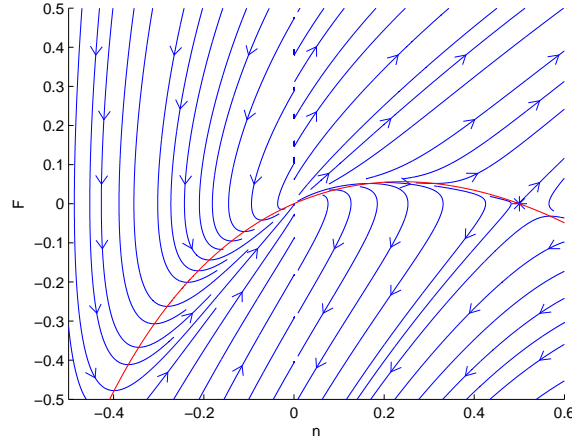


Figure 6: Phase plane of  $(n, \mathbf{F})$  with  $\tau = 0.5$  and  $b = 1.5$ ; \* is the saddle point  $(0.5, 0)$ ; the red line is the function  $\Xi(n) = -\frac{\tau(n+1-b)(2+n)n}{1+n}$ .

Obviously, if a trajectory interacts with the curve  $\Xi = \Xi(n)$ , then the interacting point is a critical point of the trajectory; and all critical points of a trajectory lie on the curve  $\Xi(n)$ . We draw the phase-plane of  $(n, \mathbf{F})$  in Figure 6 with  $\tau = 0.5$  and  $b = 1.5$ . To state our results more precisely, we need the following definition.

**Definition 2.1.** *If  $(\rho, E)$  is an interior subsonic (interior supersonic) solution to system (1.5) on an interval  $[0, L]$  satisfying  $\rho(0) = \rho(L) = 1$ , then the corresponding trajectory  $\mathbf{E} = \mathbf{E}(\rho)$  in the phase-plane  $(\rho, \mathbf{E})$  is called an interior subsonic (interior supersonic) trajectory to system (1.5). And the transformed trajectory  $\mathbf{F} = \mathbf{F}(n)$  in the  $(n, \mathbf{F})$  plane is called an interior positive (interior negative) trajectory to system (2.67).*

Clearly, an interior subsonic (interior supersonic) trajectory corresponds to an interior subsonic (interior supersonic) solution to system (1.5) on some interval. Instead of studying system (1.5) directly, we turn to analyze the structure of solutions to the transformed system (2.67). Based on the analysis of the relation between  $\mathbf{F}(n)$  and  $\Xi(n)$ , we first obtain the following important lemma.

**Lemma 2.5.** *When  $0 < \tau < \frac{1}{2\sqrt{b^3+b}}$ , all interior positive trajectories to system (2.67) start from the point  $(0, 0)$ .*

*Proof.* It is easy to see that there are two zero points of  $\Xi(n)$  on  $[0, +\infty)$ :  $n_1 = 0$ ,  $n_2 = b - 1$  and

$$\Xi'(n) = -\tau \left( 2 - b + 2n - \frac{b}{(1+n)^2} \right) \text{ for } n \geq 0, \quad (2.70)$$

$$\Xi''(n) = -2\tau(1 + \frac{b}{(1+n)^3}) < 0 \text{ for } n \geq 0, \quad (2.71)$$

$$\Xi'(0) = 2(b-1)\tau > 0 \text{ and } \Xi'(b-1) = -\tau(b - \frac{1}{b}) < 0.$$

Thus,  $\Xi(n)$  is concave on  $[0, \infty)$  and has only one maximal point denoted by  $n^*$  that only depends on  $b$ . We just focus on the region  $\mathbf{F} \geq 0$ . By (2.68) and (2.69),

$$\frac{d\mathbf{F}}{dn} = -\frac{\Xi}{\tau(1+n)^2\mathbf{F}} + \frac{1}{\tau(1+n)^2} \quad (2.72)$$

which is equivalent to

$$\frac{d\mathbf{F}}{dn} = \frac{1}{\tau(1+n)^2} \cdot \frac{\mathbf{F} - \beta\Xi}{\mathbf{F}} + \frac{(\beta-1)\Xi}{\tau(1+n)^2\mathbf{F}}, \quad (2.73)$$

where  $\beta > 0$  is a constant to be determined later. This equation in combination with (2.70) leads to

$$\begin{aligned} (\mathbf{F}^2 - \beta^2\Xi^2)' &= \frac{2(\mathbf{F} - \beta\Xi)}{\tau(1+n)^2} + 2\Xi \left[ \frac{\beta-1}{\tau(1+n)^2} + \tau\beta^2 \left( 2 - b + 2n - \frac{b}{(1+n)^2} \right) \right] \\ &= (\mathbf{F}^2 - \beta^2\Xi^2) \cdot \frac{2}{\tau(1+n)^2(\mathbf{F} + \beta\Xi)} + 2\Xi \cdot I, \end{aligned} \quad (2.74)$$

where  $I := \frac{\beta-1}{\tau(1+n)^2} + \tau\beta^2 \left( 2 - b + 2n - \frac{b}{(1+n)^2} \right)$ . Since  $\Xi(0) = 0$ , if  $\mathbf{F}(0) = h > 0$ , then we have  $\mathbf{F}^2(0) - \beta^2\Xi^2(0) = h^2 > 0$  for any  $\beta > 0$ . We next determine  $\beta$  such that  $I > 0$  for  $n \in [0, b-1]$ . To do this, we set  $\beta = \frac{c_0}{\tau^2}$  with  $c_0 = \frac{1}{2(b^3+b)}$ . When  $\tau^2 < \frac{c_0}{2}$ , we have for  $n \in [0, b-1]$

$$\begin{aligned} I &= \frac{1}{\tau(1+n)^2} \cdot \left[ \frac{c_0}{\tau^2} - 1 + \frac{c_0^2}{\tau^2} \cdot (2(1+n)^3 - b(1+n)^2 - b) \right] \\ &\geq \frac{1}{\tau(1+n)^2} \cdot \left[ \frac{c_0}{\tau^2} - 1 - \frac{c_0^2}{\tau^2} \cdot (b^3 + b) \right] \\ &= \frac{1}{\tau(1+n)^2} \cdot \left[ \frac{c_0}{\tau^2} \cdot (1 - c_0(b^3 + b)) - 1 \right] \\ &= \frac{1}{\tau(1+n)^2} \cdot \left( \frac{c_0}{2\tau^2} - 1 \right) \\ &> 0. \end{aligned}$$

Noting  $\Xi(n) > 0$  on  $(0, b-1)$ , it then follows from (2.74) that

$$\mathbf{F}^2(n) > \beta^2\Xi^2(n) \text{ for } n \in [0, b-1]. \quad (2.75)$$

Since  $(b-1, 0)$  is a saddle point lying on the curve  $\Xi = \Xi(n)$ , the trajectories starting from  $(0, h)$  with  $h > 0$  can not go back to the line  $n = 0$ , but go to infinity. Obviously, a trajectory can not start from  $(0, -h)$ . Therefore, when  $\tau < \frac{1}{2\sqrt{b^3+b}}$ , all interior positive trajectories to system (2.67) must start from  $(0, 0)$ .  $\square$

**Lemma 2.6.** *When  $0 < \tau < \frac{1}{3\sqrt{b^3+b}}$ , all interior positive trajectories to system (2.67) satisfy*

$$\mathbf{F}(n) \leq \frac{3}{2} \cdot \Xi(n) \text{ for } n \geq 0. \quad (2.76)$$

*Proof.* Taking  $\beta = \frac{3}{2}$  in (2.74), when  $\tau^2 < \frac{1}{9(b^3+b)}$ , we have for  $n \in [0, b-1]$

$$\begin{aligned} I &= \frac{1}{\tau(1+n)^2} \cdot \left[ \frac{1}{2} + \frac{9\tau^2}{4} \cdot (2(1+n)^3 - b(1+n)^2 - b) \right] \\ &\geq \frac{1}{\tau(1+n)^2} \cdot \left[ \frac{1}{2} - \frac{9\tau^2}{4} \cdot (b^3 + b) \right] \\ &> 0. \end{aligned}$$

If there is a point  $\bar{n} \in (0, b-1)$  on the trajectory such that  $\mathbf{F}(\bar{n}) > \frac{3}{2}\Xi(\bar{n})$ , then noting  $\Xi(n) > 0$  and  $I > 0$  on  $(\bar{n}, b-1)$ , we get from (2.74) that  $\mathbf{F}(n) > \frac{3}{2}\Xi(n)$  on  $(\bar{n}, b-1)$ . Because  $(b-1, 0)$  is a saddle point, this trajectory will go to infinity. We hence get (2.76).  $\square$

**Lemma 2.7.** *When  $0 < \tau < \frac{1}{2\sqrt{b^3+b}}$ , all interior positive trajectories to system (2.67) with  $\mathbf{F} \geq 0$  are Lipschitz continuous on a neighborhood of  $n = 0$ .*

*Proof.* We first present a lower bound of  $\frac{d\mathbf{F}}{dn}$ . Notice that all critical points of trajectories lie on the curve  $\Xi = \Xi(n)$ . We claim that an interior positive trajectory to system (2.67) must have at least one critical point on  $(0, b-1)$ . Otherwise, the trajectory has no critical point on  $(0, b-1)$ , then

$$\mathbf{F}'(n) > 0 \text{ on } (0, b-1) \text{ or } \mathbf{F}'(n) < 0 \text{ on } (0, b-1).$$

If the former case holds, by (2.68) and (2.69),

$$(\mathbf{F} - \Xi)'(n) > 0 \text{ on } (0, b-1).$$

By Lemma 2.5, when  $\tau < \frac{1}{2\sqrt{b^3+b}}$ , it holds  $\mathbf{F}(0) = 0 = \Xi(0)$ , then it follows that  $\mathbf{F}(n) > \Xi(n)$  on  $(0, b-1)$ . Since  $(b-1, 0)$  is a saddle point, this indicates that the trajectory can not go back to the line  $n = 0$  but goes to infinity. If the latter case holds, since  $\mathbf{F}(0) = 0$ , we get

$$\mathbf{F}(n) < 0 \text{ for any } n \in (0, b-1).$$

Using (2.68) again, noting  $n+1-b < 0$  for  $n \in (0, b-1)$ , we derive

$$\mathbf{F}'(n) > \frac{1}{\tau(1+n)^2} > 0 \text{ on } (0, b-1),$$

which is a contradiction. Thus, an interior positive trajectory to system (2.67) has at least one critical point over  $(0, b-1)$ .

We next claim that an interior positive trajectory has at most one critical point. Denote by  $n_0$  a critical point of this trajectory. Taking  $\beta = 1$  in (2.74), and using (2.70), we have

$$\begin{aligned} (\mathbf{F}^2 - \Xi^2)' &= \frac{2(\mathbf{F} - \Xi)}{\tau(1+n)^2} + 2\Xi\tau \left( 2 - b + 2n - \frac{b}{(1+n)^2} \right) \\ &= (\mathbf{F}^2 - \Xi^2) \cdot \frac{2}{\tau(1+n)^2(\mathbf{F} + \Xi)} - 2\Xi\Xi'. \end{aligned} \quad (2.77)$$

Recall  $n^*$  is the maximal point of the function  $\Xi(n)$  on  $(0, b-1)$ . If  $n_0 \geq n^*$ , noting  $\mathbf{F}(n_0) = \Xi(n_0)$  and  $\Xi(n) > 0$ ,  $\Xi'(n) < 0$  on  $(n^*, b-1)$ , it follows from (2.77) that

$$\mathbf{F}(n) > \Xi(n) \quad \text{over } (n^*, b-1).$$

Because  $(b-1, 0)$  is a saddle point, this trajectory will go to infinity. Thus,  $n_0 \in (0, n^*)$ .

Now since  $\Xi(n) > 0$ ,  $\Xi'(n) > 0$  on  $(n_0, n^*)$  and  $\mathbf{F}(n_0) = \Xi(n_0)$ , by (2.77) again, we have

$$\mathbf{F}(n) < \Xi(n) \quad \text{over } (n_0, n^*]. \quad (2.78)$$

Since all critical points of the trajectory are on the curve  $\Xi(n)$ , (2.78) indicates that there is no other critical point on  $(n_0, n^*]$  for this trajectory. On the other hand, suppose that there is a critical point  $n_1 \in (0, n_0)$  for this trajectory, then

$$\mathbf{F}(n_1) = \Xi(n_1), \quad \Xi(n) > 0 \quad \text{and} \quad \Xi'(n) > 0 \quad \text{on } (n_1, n^*].$$

Applying (2.77) repeatedly, we get

$$\mathbf{F}(n) < \Xi(n) \quad \text{for } n \in (n_1, n^*].$$

This contradicts to the fact that  $\mathbf{F}(n_0) = \Xi(n_0)$  because  $n_0 \in (n_1, n^*)$ . Thus, there is no critical point on  $(0, n_0)$  for this trajectory, and  $n_0$  is the unique critical point of this interior positive trajectory. As a consequence, we conclude that

$$\frac{d\mathbf{F}(n)}{dn} > 0 \quad \text{on } (0, n_0). \quad (2.79)$$

We next derive an upper bound of  $\frac{dF}{dn}$ . By (2.67), we get

$$F_x = n + 1 - b + \frac{n_x}{\tau(1+n)^2} = n + 1 - b - \frac{1}{\tau} \cdot \left( \frac{1}{1+n} \right)_x. \quad (2.80)$$

Noting  $n(0) = 0$ , by the continuity of the trajectory,  $0 \leq n < b-1$  on  $[0, z]$  for some  $z > 0$ .

Noting  $F(0) = 0$ , hence, for  $x \in [0, z]$

$$F(x) < -\frac{1}{\tau} \int_0^x \left( \frac{1}{1+n} \right)_x dx = \frac{1}{\tau} - \frac{1}{\tau(1+n)}.$$



It then follows that for  $n \in [0, b-1]$  and  $\mathbf{F} \geq 0$ ,

$$\frac{d\mathbf{F}(n)}{dn} < \frac{\tau(n+1-b)(2+n)}{(1+n)^2} + \frac{1}{\tau(1+n)^2} \leq \frac{1}{\tau}.$$

This estimate together with (2.79) implies the trajectory is Lipschitz continuous on  $(0, n_0)$ .  $\square$

**Lemma 2.8.** *When  $0 < \tau < \min\{\frac{1}{3\sqrt{b^3+b}}, \frac{1}{4\sqrt{b-1}}\}$ , all interior positive trajectories to system (2.67) with  $\mathbf{F} \geq 0$  are  $C^1$  smooth on a neighborhood of  $n = 0$  and*

$$\frac{d\mathbf{F}}{dn}(0) = \frac{1}{2} \left( \frac{1}{\tau} - \sqrt{\frac{1}{\tau^2} - 8(b-1)} \right). \quad (2.81)$$

*Proof.* By Lemma 2.7, we only need to show that the second order derivative of the trajectory does not change sign on a neighborhood of  $n = 0$ , i.e.

$$\frac{d^2\mathbf{F}}{dn^2} \text{ does not change sign, if } 0 < n \ll 1. \quad (2.82)$$

Step 1. We first compute  $\frac{d^2\mathbf{F}}{dn^2}$ . By (2.68) and (2.69),

$$(1+n)^2 \mathbf{F}\mathbf{F}' = \frac{1}{\tau}(\mathbf{F} - \Xi). \quad (2.83)$$

Notice that  $\mathbf{F}(n)$  is  $C^\infty$  over  $(0, b-1)$ . Differentiating (2.83) in  $n$  and using the first equality of (2.72), a direct calculation yields

$$\begin{aligned} (1+n)^2 \mathbf{F}\mathbf{F}'' &= -2(1+n)\mathbf{F}\mathbf{F}' - (1+n)^2(\mathbf{F}')^2 + \frac{1}{\tau}(\mathbf{F}' - \Xi') \\ &= -\frac{1}{\tau(1+n)\mathbf{F}^2} \left[ 2\mathbf{F}^3 - (2\Xi - (1+n)\Xi')\mathbf{F}^2 - \frac{\Xi\mathbf{F}}{\tau(1+n)} + \frac{\Xi^2}{\tau(1+n)} \right]. \end{aligned} \quad (2.84)$$

By (2.69) and (2.70), it is easy to see that

$$\begin{aligned} 2\Xi - (1+n)\Xi' &= -\frac{2\tau(n+1-b)(2+n)n}{1+n} + \tau(1+n) \left( 2-b+2n - \frac{b}{(1+n)^2} \right) \\ &= \frac{\tau}{1+n} [2(n+1-b) + bn(2+n)]. \end{aligned}$$

It then follows that

$$\begin{aligned} \mathbf{F}'' &= -\frac{2}{\tau(1+n)^3\mathbf{F}^3} \left\{ \mathbf{F}^3 - \frac{\tau[2(n+1-b) + bn(2+n)]}{2(1+n)} \cdot \mathbf{F}^2 \right. \\ &\quad \left. + \frac{(2+n)n(n+1-b)}{2(1+n)^2} \cdot \mathbf{F} + \frac{\tau(2+n)^2n^2(n+1-b)^2}{2(1+n)^3} \right\} \\ &\triangleq -\frac{2}{\tau(1+n)^3\mathbf{F}^3} \cdot H_2(n, \mathbf{F}). \end{aligned} \quad (2.85)$$

Step 2. We next solve the equation  $H_2(n, \mathbf{F}) = 0$ , which is a third order algebraic equation in the form:

$$\mathbf{F}^3 + k\mathbf{F}^2 + m\mathbf{F} + \ell = 0, \quad (2.86)$$

where

$$\begin{aligned} k &= -\frac{\tau[2(n+1-b) + bn(2+n)]}{2(1+n)}, \quad m = \frac{(2+n)n(n+1-b)}{2(1+n)^2}, \\ \ell &= \frac{\tau(2+n)^2n^2(n+1-b)^2}{2(1+n)^3}. \end{aligned} \quad (2.87)$$

Denote by  $p = -\frac{k^2}{3} + m$ ,  $q = 2(\frac{k}{3})^3 - \frac{km}{3} + \ell$ , by Cardan's formula, equation (2.86) has three roots:

$$F_1 = A^{\frac{1}{3}} + B^{\frac{1}{3}}, \quad F_2 = \varpi A^{\frac{1}{3}} + \varpi^2 B^{\frac{1}{3}}, \quad F_3 = \varpi^2 A^{\frac{1}{3}} + \varpi B^{\frac{1}{3}},$$

where  $\varpi = \frac{-1+\sqrt{3}i}{2}$ ,  $A = -\frac{q}{2} + [(\frac{q}{2})^2 + (\frac{p}{3})^3]^{\frac{1}{2}}$  and  $B = -\frac{q}{2} - [(\frac{q}{2})^2 + (\frac{p}{3})^3]^{\frac{1}{2}}$ . Furthermore, if  $(\frac{q}{2})^2 + (\frac{p}{3})^3 \leq 0$ , then all roots are real valued. We claim that when  $\tau < \frac{1}{4\sqrt{b-1}}$  and  $0 < n \ll 1$ , then  $(\frac{q}{2})^2 + (\frac{p}{3})^3 \leq 0$ . Actually, a simple calculation gives

$$\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3 = \frac{1}{4 \cdot 3^4} [(km - 9\ell)^2 - 4(k^2 - 3m)(m^2 - 3k\ell)]. \quad (2.88)$$

When  $0 < n \ll 1$ , by (2.87),

$$k = \tau(b-1) + O(n), \quad m = (1-b)n + O(n^2), \quad \ell = 2\tau(b-1)^2n^2 + O(n^3).$$

It then follows that

$$\begin{aligned} km - 9\ell &= -\tau(b-1)^2n + O(n^2), \quad k^2 - 3m = (b-1)^2\tau^2 + O(n), \\ m^2 - 3k\ell &= (b-1)^2n^2 - 6\tau^2(b-1)^3n^2 + O(n^3). \end{aligned}$$

Substituting these three estimates into (2.88) yields

$$\begin{aligned} \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3 &= \frac{1}{4 \cdot 3^4} \cdot [(b-1)^4\tau^2n^2 - 4(b-1)^2\tau^2((b-1)^2 - 6\tau^2(b-1)^3)n^2 + O(n^3)] \\ &= \frac{1}{4 \cdot 3^4} \cdot [3(b-1)^4\tau^2(-1 + 8\tau^2(b-1))n^2 + O(n^3)]. \end{aligned}$$

Thus, when  $\tau < \frac{1}{4\sqrt{b-1}}$  and  $0 < n \ll 1$ , we have  $(\frac{q}{2})^2 + (\frac{p}{3})^3 < 0$ .

Now all roots of the equation  $H_2(n, \mathbf{F}) = 0$  are real valued functions. And clearly, they are analytic in  $n$  on  $(0, b-1)$ . We then take an expansion of the roots denoted by  $\mathbf{F}_0(n)$  as

$$\mathbf{F}_0(n) = \theta_0 + \theta_1n + O(n^2),$$

and substitute this formula into  $H_2(n, \mathbf{F}) = 0$  to get

$$\theta_0 = 0 \quad \text{or} \quad \theta_0 = -\tau(b-1) < 0,$$

and

$$\theta_1 = \frac{1}{2} \left( \frac{1}{\tau} + \sqrt{\frac{1}{\tau^2} - 8(b-1)} \right) \quad \text{or} \quad \theta_1 = \frac{1}{2} \left( \frac{1}{\tau} - \sqrt{\frac{1}{\tau^2} - 8(b-1)} \right).$$

Notice that when  $\tau \ll 1$ ,  $\frac{1}{2} \left( \frac{1}{\tau} + \sqrt{\frac{1}{\tau^2} - 8(b-1)} \right) = O(\frac{1}{\tau})$  and  $\frac{1}{2} \left( \frac{1}{\tau} - \sqrt{\frac{1}{\tau^2} - 8(b-1)} \right) = \frac{4(b-1)\tau}{1 + \sqrt{1 - 8(b-1)\tau^2}} = O(\tau)$ . Because we are interested in the interior positive trajectories with  $\mathbf{F} \geq 0$ , by Lemma 2.6, it holds that

$$\theta_0 = 0 \quad \text{and} \quad \theta_1 = \frac{1}{2} \left( \frac{1}{\tau} - \sqrt{\frac{1}{\tau^2} - 8(b-1)} \right). \quad (2.89)$$

Thus, the solution curve of the equation  $H_2(n, \mathbf{F}) = 0$  satisfies

$$\mathbf{F}_0(n) = \theta_1 n + O(n^2) = \frac{4(b-1)\tau}{1 + \sqrt{1 - 8(b-1)\tau^2}} \cdot n + O(n^2). \quad (2.90)$$

Step 3. We proceed to show that when  $0 < n \ll 1$ , the function  $\frac{d\mathbf{F}}{dn}(n)$  is monotone.

Assume that  $\hat{n}_0 > 0$  is a critical point of the function  $\frac{d\mathbf{F}}{dn}(n)$ , then  $\frac{d^2\mathbf{F}}{dn^2}(\hat{n}_0) = 0$ . We claim that when  $\hat{n}_0$  is small enough, it holds that  $\frac{d^3\mathbf{F}}{dn^3}(\hat{n}_0) > 0$ . Differentiating (2.84) in  $n$ , we have

$$\begin{aligned} & 2(1+n)\mathbf{F}\mathbf{F}'' + (1+n)^2\mathbf{F}'\mathbf{F}'' + (1+n)^2\mathbf{F}\mathbf{F}''' \\ &= -2\mathbf{F}\mathbf{F}' - 4(1+n)(\mathbf{F}')^2 - 2(1+n)\mathbf{F}\mathbf{F}'' - 2(1+n)^2\mathbf{F}'\mathbf{F}'' + \frac{1}{\tau}(\mathbf{F}'' - \Xi''). \end{aligned}$$

Noting  $\mathbf{F}''(\hat{n}_0) = 0$ , it then follows from (2.71) that

$$\begin{aligned} (1 + \hat{n}_0)^2\mathbf{F}\mathbf{F}'''(\hat{n}_0) &= -2\mathbf{F}\mathbf{F}'(\hat{n}_0) - 4(1 + \hat{n}_0)(\mathbf{F}'(\hat{n}_0))^2 - \frac{\Xi''(\hat{n}_0)}{\tau} \\ &= -2\mathbf{F}\mathbf{F}'(\hat{n}_0) - 4(1 + \hat{n}_0)(\mathbf{F}'(\hat{n}_0))^2 + 2 + \frac{2b}{(1 + \hat{n}_0)^3}. \end{aligned} \quad (2.91)$$

Using (2.84) again, since  $\mathbf{F}''(\hat{n}_0) = 0$ , it holds

$$(1 + \hat{n}_0)^2(\mathbf{F}'(\hat{n}_0))^2 = -2(1 + \hat{n}_0)\mathbf{F}\mathbf{F}'(\hat{n}_0) + \frac{1}{\tau}(\mathbf{F}'(\hat{n}_0) - \Xi'(\hat{n}_0)).$$

Substituting this inequality into (2.91) and using (2.72) leads to

$$\begin{aligned} & (1 + \hat{n}_0)^3\mathbf{F}\mathbf{F}'''(\hat{n}_0) \\ &= -2(1 + \hat{n}_0)\mathbf{F}\mathbf{F}'(\hat{n}_0) - 4(1 + \hat{n}_0)^2(\mathbf{F}'(\hat{n}_0))^2 + 2(1 + \hat{n}_0) + \frac{2b}{(1 + \hat{n}_0)^2} \\ &= 6(1 + \hat{n}_0)\mathbf{F}\mathbf{F}'(\hat{n}_0) - \frac{4\mathbf{F}'(\hat{n}_0)}{\tau} + \frac{4\Xi'(\hat{n}_0)}{\tau} + 2(1 + \hat{n}_0) + \frac{2b}{(1 + \hat{n}_0)^2} \\ &= \frac{6(\mathbf{F}(\hat{n}_0) - \Xi(\hat{n}_0))}{\tau(1 + \hat{n}_0)} - \frac{4(\mathbf{F}(\hat{n}_0) - \Xi(\hat{n}_0))}{\tau^2(1 + \hat{n}_0)^2\mathbf{F}(\hat{n}_0)} + \frac{4\Xi'(\hat{n}_0)}{\tau} + 2(1 + \hat{n}_0) + \frac{2b}{(1 + \hat{n}_0)^2} \\ &= \frac{2}{(1 + \hat{n}_0)^2\mathbf{F}} \cdot \left[ \frac{3\mathbf{F}(\mathbf{F} - \Xi)(1 + \hat{n}_0)}{\tau} - \frac{2(\mathbf{F} - \Xi)}{\tau^2} + \frac{2(1 + \hat{n}_0)^2\mathbf{F}\Xi'}{\tau} + (1 + \hat{n}_0)^3\mathbf{F} + b\mathbf{F} \right] \\ &:= \frac{2}{(1 + \hat{n}_0)^2\mathbf{F}} \cdot J(\hat{n}_0). \end{aligned} \quad (2.92)$$

By (2.69), (2.70) and (2.90), when  $\hat{n}_0 \ll 1$ ,

$$\begin{aligned}\mathbf{F}(\hat{n}_0) &= \theta_1 \hat{n}_0 + O(\hat{n}_0^2), \quad \mathbf{F}(\hat{n}_0) - \Xi(\hat{n}_0) = (\theta_1 - 2\tau(b-1))\hat{n}_0 + O(\hat{n}_0^2), \\ \hat{\Xi}'(\hat{n}_0) &= 2\tau(b-1) + O(\hat{n}_0).\end{aligned}$$

Thus,

$$\begin{aligned}J(\hat{n}_0) &= -\frac{2(\theta_1 - 2\tau(b-1))}{\tau^2} \hat{n}_0 + 4(b-1)\theta_1 \hat{n}_0 + \theta_1 \hat{n}_0 + b\theta_1 \hat{n}_0 + O(\hat{n}_0^2) \\ &= \left[ \frac{4(b-1)}{\tau} - \frac{2\theta_1}{\tau^2} + (5b-3)\theta_1 \right] \hat{n}_0 + O(\hat{n}_0^2).\end{aligned}\tag{2.93}$$

By (2.89),

$$\frac{4(b-1)}{\tau} - \frac{2\theta_1}{\tau^2} = -\theta_1 \cdot \frac{8(b-1)}{1 + \sqrt{1 - 8(b-1)\tau^2}}.$$

It hence follows that if  $\tau \ll 1$  such that  $\tau^2 < \frac{1}{16(b-1)} < \frac{2}{25(b-1)}$ , then

$$\begin{aligned}\frac{4(b-1)}{\tau} - \frac{2\theta_1}{\tau^2} + (5b-3)\theta_1 &= \theta_1 \left[ 5b-3 - \frac{8(b-1)}{1 + \sqrt{1 - 8(b-1)\tau^2}} \right] \\ &> \theta_1(5b-3-5(b-1)) \\ &= 2\theta_1 > 0.\end{aligned}$$

Substituting this inequality into (2.93) and then (2.92), we conclude that

$$\mathbf{F}'''(\hat{n}_0) > 0 \quad \text{if } \hat{n}_0 \ll 1 \quad \text{and } \tau < \frac{1}{4\sqrt{b-1}}.$$

Thus, the critical point  $\hat{n}_0$  must be the local minimal point of  $\frac{d\mathbf{F}}{dn}(n)$ . And hence there exists  $n_2 > 0$  such that the function  $\frac{d\mathbf{F}}{dn}(n)$  has at most one critical point over  $(0, n_2)$ . This implies  $\frac{d^2\mathbf{F}}{dn^2}$  could change sign at most once on  $(0, n_2)$ . As a consequence, there exists  $n_3 \in (0, n_2)$  such that the function  $\frac{d\mathbf{F}}{dn}(n)$  is monotone on  $(0, n_3)$ .

Step 4. Now by Lemma 2.7 and the monotonicity of  $\frac{d\mathbf{F}}{dn}$ , one can easily see that

$$\lim_{n \rightarrow 0^+} \mathbf{F}'(n) \text{ exists } \triangleq \mathbf{F}'(0).$$

Then  $\mathbf{F}'(n)$  is continuous on  $[0, n_2]$ . It is left to show (2.81). Applying L'Hospital principle to equation (2.68) at  $n = 0$ , it holds that

$$\mathbf{F}'(0) = \frac{2(1-b)}{\mathbf{F}'(0)} + \frac{1}{\tau}.$$

Thus,

$$\mathbf{F}'(0) = \frac{1}{2} \left( \frac{1}{\tau} + \sqrt{\frac{1}{\tau^2} - 8(b-1)} \right) = O\left(\frac{1}{\tau}\right) \quad \text{or} \quad \mathbf{F}'(0) = \frac{1}{2} \left( \frac{1}{\tau} - \sqrt{\frac{1}{\tau^2} - 8(b-1)} \right) = O(\tau).$$

By Lemma 2.6,  $\mathbf{F}'(0) = \frac{1}{2} \left( \frac{1}{\tau} - \sqrt{\frac{1}{\tau^2} - 8(b-1)} \right)$ . □

**Theorem 2.4.** *Assume that  $b(x) = b > 1$  is a constant. There exists a constant  $\tau_0 = \tau_0(b)$  only depending on  $b$ , such that for any  $0 < \tau < \tau_0$  the interior subsonic solution to system (1.5) satisfies*

$$\rho \in C^1[0, \epsilon], \quad \rho(0) = 1, \quad E(0) = \frac{1}{\tau} \quad \text{and} \quad \rho_x(0) = \frac{1}{4} \left( \frac{1}{\tau} - \sqrt{\frac{1}{\tau^2} - 8(b-1)} \right) \quad (2.94)$$

for some  $\epsilon > 0$ .

*Proof.* Recalling the transformation (2.66),  $\rho = n + 1$  and  $E = F + \frac{1}{\tau(n+1)}$ . By Lemma 2.5, one can find that all interior subsonic trajectories to system (1.5) must start from  $(1, \frac{1}{\tau})$ . In other words, all interior subsonic solutions to system (1.5) must satisfy  $\rho(0) = 1$  and  $E(0) = \frac{1}{\tau}$ . By L'Hospital principle and (2.81),

$$\lim_{n \rightarrow 0^+} \frac{n}{\mathbf{F}(n)} = \lim_{n \rightarrow 0^+} \frac{1}{\mathbf{F}'(n)} = \frac{1}{\theta_1},$$

which together with the first equation of (2.67) gives

$$n_x(0) = \lim_{x \rightarrow 0^+} \frac{F(x)}{2n(x)} = \frac{\theta_1}{2}.$$

Thus,  $n \in C^1[0, \epsilon]$  for some  $\epsilon > 0$ . Recalling  $n = \rho - 1$ , we have  $\rho \in C^1[0, \epsilon]$  for some  $\epsilon > 0$ , and

$$\rho_x(0) = \frac{\theta_1}{2} = \frac{1}{4} \left( \frac{1}{\tau} - \sqrt{\frac{1}{\tau^2} - 8(b-1)} \right),$$

where we have used  $\theta_1 = \frac{1}{2} \left( \frac{1}{\tau} - \sqrt{\frac{1}{\tau^2} - 8(b-1)} \right)$ . □

We next study the structure of the interior supersonic solutions. To do so, we still study the transformed equations (2.67) and (2.68) but with  $n \in (-1, 0]$ .

**Lemma 2.9.** *When  $0 < \tau < \frac{1}{3\sqrt{b}}$ , all interior negative trajectories to system (2.67) end at the point  $(0, 0)$ .*

*Proof.* By (2.69)-(2.71),

$$\Xi(n) \leq 0, \quad \Xi'(n) > 2\tau(b-1) > 0, \quad \Xi''(n) < 0 \quad \text{for } n \in (-1, 0], \quad (2.95)$$

$$\lim_{n \rightarrow -1} \Xi(n) = -\infty. \quad (2.96)$$

We next focus on the region  $\mathbf{F} \leq 0$ . Notice that (2.74) still holds. If  $\mathbf{F}(n) = -h < 0$ , then  $\mathbf{F}^2(0) - \beta^2 \Xi^2(0) = h^2 > 0$  for any  $\beta > 0$ . To ensure  $\mathbf{F}^2(n) > \beta^2 \Xi^2(n)$  for any  $n \in (-1, 0]$ , we

also need to determine  $\beta$  such that  $I > 0$  for  $n \in (-1, 0]$ . Setting  $\beta = \frac{c_1}{\tau^2}$  with  $c_1 = \frac{1}{3b}$ , when  $\tau < \frac{1}{3\sqrt{b}}$ , we have for  $n \in (-1, 0]$

$$\begin{aligned} I &= \frac{1}{\tau(1+n)^2} \cdot \left[ \frac{c_1}{\tau^2} - 1 + \frac{c_1^2}{\tau^2} \cdot (2(1+n)^3 - b(1+n)^2 - b) \right] \\ &\geq \frac{1}{\tau(1+n)^2} \cdot \left[ \frac{c_1}{\tau^2} - 1 - \frac{2bc_1^2}{\tau^2} \right] \\ &= \frac{1}{\tau(1+n)^2} \cdot \left( \frac{1}{9b\tau^2} - 1 \right) \\ &> 0. \end{aligned}$$

It then follows from (2.74) that  $\mathbf{F}^2(n) > \frac{\Xi^2(n)}{3b\tau^2}$  for  $n \in (-1, 0)$ . Noting  $\mathbf{F}(0) < 0$  and  $\Xi(n) < 0$  on  $(-1, 0)$ , we get

$$\mathbf{F}(n) < \Xi(n) \quad \text{for } n \in (-1, 0).$$

It hence follows from (2.96) that

$$\lim_{n \rightarrow -1} \mathbf{F}(n) = -\infty,$$

and the trajectory ending at  $(0, -h)$  with  $h > 0$  does not start from a point of the line  $n = 0$ . Thus, when  $\tau < \frac{1}{3\sqrt{b}}$ , all interior negative trajectories should end at the point  $(0, 0)$ .  $\square$

**Lemma 2.10.** *When  $\tau < \frac{1}{3\sqrt{b}}$ , all interior negative trajectories to system (2.67) satisfy*

$$\mathbf{F}(n) \geq \frac{3}{2} \cdot \Xi(n) \quad \text{for } n \in (-1, 0]. \quad (2.97)$$

*Proof.* Taking  $\beta = \frac{3}{2}$  in (2.74), when  $\tau < \frac{1}{3\sqrt{b}}$ , we have for  $n \in (-1, 0]$

$$\begin{aligned} I &= \frac{1}{\tau(1+n)^2} \cdot \left[ \frac{1}{2} + \frac{9\tau^2}{4} \cdot (2(1+n)^3 - b(1+n)^2 - b) \right] \\ &\geq \frac{1}{\tau(1+n)^2} \cdot \left[ \frac{1}{2} - \frac{9\tau^2}{4} \cdot 2b \right] \\ &> 0. \end{aligned}$$

If there is a point  $\bar{n} \in (-1, 0)$  on the trajectory such that  $\mathbf{F}(\bar{n}) < \frac{3}{2} \cdot \Xi(\bar{n}) < 0$ , then noting  $\Xi(n) < 0$  and  $I > 0$  on  $(-1, \bar{n})$ , by (2.74), we have

$$\mathbf{F}^2(n) > \frac{9}{4} \cdot \Xi^2(n) \quad \text{on } (-1, \bar{n}).$$

Because  $\Xi(\bar{n}) < 0$  and  $\mathbf{F}(\bar{n}) < 0$  on  $(-1, \bar{n})$ , we have  $\mathbf{F}(n) < \frac{3}{2} \cdot \Xi(n)$  for  $n \in (-1, \bar{n})$ . Thus, by (2.96),  $\lim_{n \rightarrow -1} \mathbf{F}(n) = -\infty$ , and this trajectory starts from infinity and can not be an interior negative trajectory to system (2.67). We hence get (2.97).  $\square$

**Lemma 2.11.** *When  $\tau < \frac{1}{3\sqrt{b}}$ , all interior negative trajectories to system (2.67) with  $\mathbf{F} \leq 0$  are Lipschitz continuous on a neighborhood of  $n = 0$ .*

*Proof.* We first show that an interior negative trajectory must have at least one critical point on  $(-1, 0)$ . Otherwise, the trajectory has no critical point over  $(-1, 0)$ , then

$$\mathbf{F}'(n) > 0 \text{ on } (-1, 0) \text{ or } \mathbf{F}'(n) < 0 \text{ on } (-1, 0).$$

If  $\mathbf{F}'(n) > 0$  on  $(-1, 0)$ , then

$$(\mathbf{F} - \Xi)'(n) > 0 \text{ on } (-1, 0).$$

By Lemma 2.9, when  $\tau < \frac{1}{3\sqrt{b}}$ ,  $\mathbf{F}(0) = \Xi(0) = 0$ , we then have  $\mathbf{F}(n) < \Xi(n) < 0$  on  $(-1, 0)$ . Thus, by (2.96)

$$\lim_{n \rightarrow -1} \mathbf{F}(n) < \lim_{n \rightarrow -1} \Xi(n) = -\infty.$$

This implies the trajectory can not start from a point of the line  $n = 0$ . If  $\mathbf{F}'(n) < 0$  on  $(-1, 0)$ , noting  $\mathbf{F}(0) = 0$ , it holds that  $\mathbf{F}(n) > 0$  for  $n \in (-1, 0)$ . By (2.68), since  $\frac{(n+1-b)(2-n)n}{(1+n)^3 F} > 0$ , we have

$$\mathbf{F}'(n) > \frac{1}{\tau(1+n)^2} > 0 \text{ on } (-1, 0),$$

which is a contradiction. Thus, an interior negative trajectory to system (2.67) has at least one critical point on  $(-1, 0)$ .

We next claim that this interior negative trajectory has at most one critical point. Denote by  $\widetilde{n}_0 \in (-1, 0)$  a critical point of this trajectory. Then by (2.95),

$$\mathbf{F}(\widetilde{n}_0) = \Xi(\widetilde{n}_0) < 0, \quad \Xi(n) < 0, \quad \Xi'(n) > 0 \text{ on } (-1, \widetilde{n}_0). \quad (2.98)$$

Noting (2.77) still holds, it follows from (2.98) and (2.77) that when  $F \leq 0$ ,

$$0 \geq \mathbf{F}(n) > \Xi(n) \text{ for } n \in (n_*, \widetilde{n}_0),$$

where  $n_*$  is the point that  $\mathbf{F}(n_*) = 0$ . In other words, there is no critical point on  $(-1, \widetilde{n}_0)$ . On the other hand, if there is a critical point  $\widetilde{n}_1 \in (\widetilde{n}_0, 0)$ , then

$$\mathbf{F}(\widetilde{n}_1) = \Xi(\widetilde{n}_1) < 0, \quad \Xi(n) < 0 \text{ and } \Xi'(n) > 0 \text{ on } (\widetilde{n}_0, \widetilde{n}_1).$$

Applying (2.77) again, we have

$$\mathbf{F}(n) > \Xi(n) \text{ for } n \in (n_*, \widetilde{n}_1),$$

which contradicts to the fact that  $\mathbf{F}(\widetilde{n}_0) = \Xi(\widetilde{n}_0)$ . Thus, there is no critical point on  $(\widetilde{n}_0, 0)$  for this trajectory, and  $\widetilde{n}_0$  is the unique critical point of this trajectory. As a consequence, we obtain

$$\frac{d\mathbf{F}(n)}{dn} > 0 \text{ on } (\widetilde{n}_0, 0). \quad (2.99)$$

We next derive an upper bound of  $\frac{d\mathbf{F}}{dn}$ . Integrating (2.80) on  $(x, 1)$ , noting  $F(1) = 0$ , we have

$$F(x) > \frac{1}{\tau} \int_x^1 \left( \frac{1}{1+n} \right) dx = \frac{1}{\tau} - \frac{1}{\tau(1+n)} = \frac{n}{\tau(1+n)}.$$

Noting  $F < 0$  on  $(x, 1)$ , it follows that  $\frac{n}{F} > \tau(1+n)$ . By (2.68), we obtain

$$\frac{d\mathbf{F}(n)}{dn} < \frac{\tau(n+1-b)(2+n)}{(1+n)^2} + \frac{1}{\tau(1+n)^2} \leq \frac{1}{\tau(1+n)^2} < \frac{1}{\tau(1+\widetilde{n}_0)^2} \quad \text{for } n \in (\widetilde{n}_0, 0).$$

This estimate together with (2.99) implies the trajectory is Lipschitz continuous on  $(\widetilde{n}_0, 0)$ .  $\square$

**Lemma 2.12.** *When  $\tau < \min\{\frac{1}{3\sqrt{b}}, \frac{1}{4\sqrt{b-1}}\}$ , all interior negative trajectories to system (2.67) with  $\mathbf{F} \leq 0$  are  $C^1$  smooth on a neighborhood of  $n = 0$  and*

$$\frac{d\mathbf{F}}{dn}(0) = \frac{1}{2} \left( \frac{1}{\tau} - \sqrt{\frac{1}{\tau^2} - 8(b-1)} \right).$$

*Proof.* The proof is quite similar to that of Lemma 2.8. The main difference is that, now the unique critical point of the function  $\frac{d\mathbf{F}}{dn}$  is the maximal point of  $\frac{d\mathbf{F}}{dn}$ . Other changes are obvious.  $\square$

On the base of Lemma 2.12, analog to Theorem 2.4, one can obtain the refined structure of the interior supersonic solution established in Theorem 2.2.

**Theorem 2.5.** *Assume that  $b(x) = b > 1$  is a constant. There exists a constant  $\tau_0 = \tau_0(b)$  such that for any  $0 < \tau < \tau_0$  the interior supersonic solution  $(\rho, E)$  on an interval  $[0, L]$  satisfies*

$$\rho \in C^1[L-\epsilon, L], \quad \rho(0) = \rho(L) = 1, \quad E(L) = \frac{1}{\tau} \quad \text{and} \quad \rho_x(L) = \frac{1}{4} \left( \frac{1}{\tau} - \sqrt{\frac{1}{\tau^2} - 8(b-1)} \right). \quad (2.100)$$

for some  $\epsilon > 0$ .

On the base of Theorems 2.4 and 2.5, we are able to construct interior  $C^1$  smooth transonic solutions to system (1.5).

**Theorem 2.6.** *Assume that  $b(x) = b > 1$  is a constant. There exists a constant  $\tau_0 = \tau_0(b)$  such that for any  $0 < \tau < \tau_0$ , there exist infinitely many interior  $C^1$  smooth transonic solutions to system (1.5)-(1.6) in the form*

$$\rho(x) = \begin{cases} \rho_{sup}(x), & x \in (0, x_0), \\ \rho_{sub}(x), & x \in (x_0, 1), \end{cases}$$

where  $x_0 \in (0, 1)$  is the location of transition,  $0 < \rho_{sup}(x) \leq 1$  and  $\rho_{sub}(x) \geq 1$  satisfy

$$\rho_{sup}(x_0) = \rho_{sub}(x_0) = 1, \quad (2.101)$$



$$\begin{aligned}
(\rho_{sup})_x(x_0) &= (\rho_{sub})_x(x_0) = \frac{1}{4} \left( \frac{1}{\tau} - \sqrt{\frac{1}{\tau^2} - 8(b-1)} \right), \\
E_{sup}(x_0) &= E_{sub}(x_0) = \frac{1}{\tau}.
\end{aligned} \tag{2.102}$$

*Proof.* For any  $x_0 \in (0, 1)$ , by Theorem 2.2, system (1.5) admits an interior supersonic solution  $\rho_{sup}$  on  $[0, x_0]$  satisfying

$$\rho_{sup}(0) = \rho_{sup}(x_0) = 1.$$

By Theorem 2.5, there exists a constant  $\tau_0 = \tau_0(b)$  such that for any  $0 < \tau < \tau_0$

$$\rho_{sup} \in C^1[x_0 - \epsilon_0, x_0], \quad E_{sup}(x_0) = \frac{1}{\tau} \quad \text{and} \quad (\rho_{sup})_x(x_0) = \frac{1}{4} \left( \frac{1}{\tau} - \sqrt{\frac{1}{\tau^2} - 8(b-1)} \right). \tag{2.103}$$

for some  $\epsilon_0 > 0$ .

On the other hand, by Theorem 2.1, system (1.5) has a unique interior subsonic solution  $\rho_{sub}$  on  $[x_0, 1]$  satisfying

$$\rho_{sub}(x_0) = \rho_{sub}(1) = 1.$$

By Theorem 2.4, there exists a constant  $\tau_1 = \tau_1(b)$  such that for any  $0 < \tau < \tau_1$

$$\rho_{sub} \in C^1[x_0, x_0 + \epsilon_1], \quad E_{sub}(x_0) = \frac{1}{\tau} \quad \text{and} \quad (\rho_{sub})_x(x_0) = \frac{1}{4} \left( \frac{1}{\tau} - \sqrt{\frac{1}{\tau^2} - 8(b-1)} \right). \tag{2.104}$$

for some  $\epsilon_1 > 0$ . We can now construct an interior  $C^1$  smooth transonic solution by

$$\rho(x) = \begin{cases} \rho_{sup}(x), & x \in [0, x_0], \\ \rho_{sub}(x), & x \in [x_0, 1]. \end{cases}$$

Furthermore, (2.101) and (2.102) follows from (2.103) and (2.104). Because  $x_0 \in (0, 1)$  is arbitrary, the  $C^1$  smooth transonic solutions are infinitely many.  $\square$

As a byproduct, one can easily see that when  $0 < \tau \ll 1$ , there is no transonic solution with shock. In other words, when  $\tau$  is small, system (1.5) admits transonic solution of  $C^1$  smooth type only.

**Theorem 2.7.** *Assume that  $b(x) = b > 1$  is a constant. There exists a constant  $\tau_0 = \tau_0(b)$  such that for any  $0 < \tau < \tau_0$ , system (1.5)-(1.6) has no transonic shock solution.*

*Proof.* We argue by contradiction. Assume that there is a transonic solution with shock. Denote by  $x_0$  the jump location. By the Rankine-Hugoniot condition (1.11) and (1.12),

$$E_l = E_r \quad \text{and} \quad \rho_l \rho_r = 1. \tag{2.105}$$

Because the solution is discontinuous, it holds  $0 < \rho_l < 1 < \rho_r$ . Clearly, there are two cases for the value of  $E_l$ :

$$E_l \leq \frac{1}{\tau} \quad \text{or} \quad E_l > \frac{1}{\tau}.$$

If the former case holds, observing that at  $x_0$ ,  $\rho_{sup}(x_0)E_{sup}(x_0) - \frac{1}{\tau} = \rho_l E_l - \frac{1}{\tau} < 0$ , it follows from the first equation of (1.5) that

$$\rho_{sup}(x_0) = \frac{\rho_l E_l - \frac{1}{\tau}}{1 - \frac{1}{\rho_l^2}} > 0.$$

Thus, we can extend this supersonic solution to an interval  $[0, L]$  such that

$$\rho_{sup}(L) = 1, \quad \text{and} \quad E_{sup}(L) < E_{sup}(x_0) = E_l < \frac{1}{\tau}.$$

Here we have used the fact that  $E_{sup}$  is monotone decreasing. Recalling the transformation (2.66), this implies

$$F_{sup}(L) = E_{sup}(L) - \frac{1}{\tau} < 0.$$

In view of the proof of Lemma 2.9, we find that the corresponding trajectory satisfies

$$\lim_{n \rightarrow -1^+} \mathbf{F}_{sup}(n) = -\infty.$$

Thus, this supersonic solution can not satisfy the left boundary condition  $\rho_{sup}(0) = 1$ , which is a contradiction.

If the latter case happen, by (2.105), we get

$$E_r > \frac{1}{\tau} \quad \text{and} \quad \rho_r > 1.$$

Thus, we can extend backward this subsonic part to an interior subsonic solution of system (1.5), still denoted by  $(\rho_{sub}, E_{sub})$  such that for some  $x_{-1} \in \mathbb{R}$

$$\rho_{sub}(x_{-1}) = 1, \quad E_{sub}(x_{-1}) > E_r > \frac{1}{\tau},$$

where we have used the fact that  $E_{sub}$  is monotone decreasing. Recalling the transformation again, we have

$$F_{sub}(x_{-1}) = E_{sub}(x_{-1}) - \frac{1}{\tau} > E_r - \frac{1}{\tau} > 0.$$

In view of the proof of Lemma 2.5, one can see that the corresponding trajectory will go to infinity, which contradicts to the right boundary condition  $\rho_{sub}(1) = 1$ . Therefore, there is no transonic solution with shock.  $\square$

*Proof of Theorem 1.1.* Combining Theorems 2.1, 2.2, 2.3, 2.6 and 2.7, we immediately obtain Theorem 1.1.  $\square$

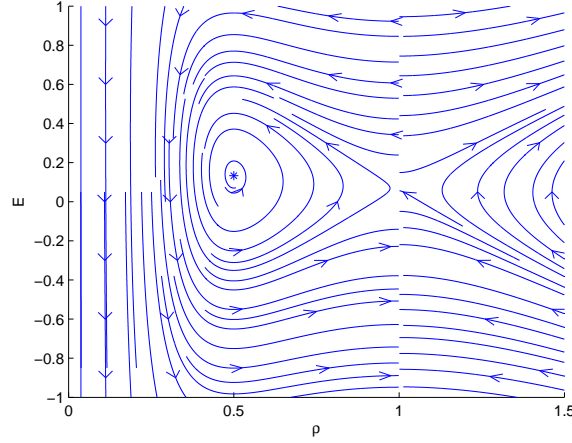


Figure 7: Phase plane of  $(\rho, E)$  with  $\tau = 15$  and  $b = 0.5$ ; \* is the focal point  $A = (0.5, 2/15)$ .

### 3 The case of supersonic doping profile

In this section, we consider the more general case of non-subsonic doping file, namely, we assume that  $0 \leq \underline{b} \leq b(x) \leq \bar{b} \leq 1$ , which allows  $b(x)$  partially supersonic and partially sonic in the domain  $[0, 1]$ . To observe the structure of the stationary solutions, let us first test the specially case with a constant supersonic doping profile  $b(x) \equiv b < 1$ , where the phase-plane analysis is helpful. In this case, the critical point  $A = (b, \frac{1}{\tau b})$  sits at the left hand side of the sonic line  $\rho = 1$ . By (2.1), the eigenvalues of the Jacobian matrix  $J(A)$  satisfy

$$\lambda_1 + \lambda_2 < 0, \quad \lambda_1 \lambda_2 > 0.$$

Thus,  $\lambda_1, \lambda_2 < 0$ , which indicates that  $A$  is a stable focal point. In view of (2.2) and (1.5), we draw the phase-plane of  $(\rho, E)$  in Figure 7 with  $\tau = 15$  and  $b = 0.5$ . From Figure 7, we see that one outside curve starts from the sonic line, passes through the supersonic regime, then ends at the sonic line, so there is a possible interior supersonic solution. The other curve starts from the sonic line, but rotates in the supersonic regime, and never ends at the sonic line, thus such a curve is not a solution. Obviously, there is no interior subsonic solution.

Now, for the general case of supersonic doping profile, we are going to prove that, there is no interior subsonic solution, nor transonic solution, even no interior supersonic solution if the doping profile  $b(x) \ll 1$  or  $\tau \ll 1$ , namely, when the semiconductor device is almost pure, or the relaxation time is really small (equivalently, the semiconductor effect is large). The supersonic solution and transonic solution exist only when the doping profile is close to the sonic line and  $\tau$  is large enough. This is totally different from the previous studies [21, 22] for the case without

semiconductor effect.

Now we are going to prove each case stated in Theorem 1.2.

### 3.1 Non-existence of interior subsonic/supersonic/transonic solutions

In this subsection, we are going to prove the non-existence of interior subsonic/supersonic/transonic solutions when the doping profile is small or the relaxation time is small.

**Theorem 3.1.** *No interior subsonic solution to (1.7) exists for the case of non-subsonic doping profile  $0 \leq \underline{b} \leq b(x) \leq \bar{b} \leq 1$ .*

*Proof.* Suppose there is an interior subsonic solution  $\rho_{sub}$  of (1.7) defined in Definition 1.1, let us take the test function by  $\varphi = (\rho - 1)^2 \in H_0^1(0, 1)$  in (1.8), then we have

$$\frac{1}{2} \int_0^1 \frac{\rho + 1}{\rho^3} \cdot |[(\rho - 1)^2]_x|^2 dx + \int_0^1 \frac{[(\rho - 1)^2]_x}{\tau \rho} dx + \int_0^1 (\rho - b)(\rho - 1)^2 dx = 0. \quad (3.1)$$

Noting that

$$\int_0^1 \frac{[(\rho - 1)^2]_x}{\tau \rho} dx = \frac{2}{\tau} \int_0^1 (\rho - \ln \rho)_x dx = 0,$$

and that  $\rho - b > 0$  on  $(0, 1)$ , namely,

$$\int_0^1 (\rho - b)(\rho - 1)^2 dx > 0,$$

then, from (3.1), we get a contradiction:

$$\frac{1}{2} \int_0^1 \frac{\rho + 1}{\rho^3} \cdot |[(\rho - 1)^2]_x|^2 dx < 0.$$

Therefore, there is no interior subsonic solution.  $\square$

**Theorem 3.2.** *No interior supersonic solution to (1.7) exists, when the doping profile  $b(x)$  is small such that  $\bar{b}(1 + \sqrt{2\bar{b}}) < 1$ , or the relaxation time  $\tau$  is small such that  $\tau < \frac{1}{3}$ .*

*Proof.* Assume that  $\rho(x)$  is an interior supersonic solution of (1.7) satisfying Definition 1.1. The velocity  $u(x) = \frac{1}{\rho(x)}$  satisfies

$$\begin{cases} \left(u - \frac{1}{u}\right) u_x = E - \frac{u}{\tau}, \\ E_x = \frac{1}{u} - b(x). \end{cases} \quad (3.2)$$

Because  $u \in C[0, 1]$ , there exists a maximal point denote by  $\hat{y}$  such that  $u(x) \leq u(\hat{y})$  for any  $x \in [0, 1]$ . At  $\hat{y}$ , the first equation of (3.2) gives

$$E(\hat{y}) = \frac{u(\hat{y})}{\tau}. \quad (3.3)$$

Multiplying the first equation of (3.2) by  $((u - 1)^2)_x$ , integrating the resultant equation over  $(\hat{y}, 1)$ , using the second equation of (3.2), and noting

$$u((u - 1)^2)_x = \frac{1}{3}((u - 1)^2(2u + 1))_x,$$

we obtain

$$\begin{aligned} & \int_{\hat{y}}^1 \frac{u(x) + 1}{2u(x)} |[(u(x) - 1)^2]_x|^2 dx \\ &= \int_{\hat{y}}^1 \left( b(x) - \frac{1}{u(x)} \right) (u(x) - 1)^2 dx - (u(\hat{y}) - 1)^2 \left( E(\hat{y}) - \frac{2u(\hat{y}) + 1}{3\tau} \right) \\ &= \int_{\hat{y}}^1 \left( b(x) - \frac{1}{u(x)} \right) (u(x) - 1)^2 dx - \frac{(u(\hat{y}) - 1)^3}{3\tau}, \end{aligned} \quad (3.4)$$

where we have used (3.3) in the second equality.

In the case  $b(x) \ll 1$ , since  $u(\hat{y}) > 1$ , we get from (3.4) that

$$\begin{aligned} & \int_{\hat{y}}^1 \frac{u(x) + 1}{2u(x)} |[(u(x) - 1)^2]_x|^2 dx \\ & \leq \int_{\hat{y}}^1 \left( b(x) - \frac{1}{u(x)} \right) (u(x) - 1)^2 dx \\ & \leq \int_{\hat{y}}^1 b(x) (u(x) - 1)^2 dx \\ & \leq \frac{1}{4} \int_{\hat{y}}^1 (u(x) - 1)^4 dx + \int_{\hat{y}}^1 b^2(x) dx \\ & \leq \frac{1}{4} \int_{\hat{y}}^1 |[(u(x) - 1)^2]_x|^2 dx + \bar{b}^2. \end{aligned} \quad (3.5)$$

Here we have used

$$\int_y^1 (u(x) - 1)^4 dx \leq \int_y^1 |[(u(x) - 1)^2]_x|^2 dx \quad \text{for } y \in (0, 1). \quad (3.6)$$

Then (3.5) gives

$$\int_y^1 |[(u(x) - 1)^2]_x|^2 dx \leq 4\bar{b}^2,$$

and further

$$u(x) \leq 1 + (2\bar{b})^{1/2} \quad \text{on } [\hat{y}, 1].$$

It then follows that

$$\left( b(x) - \frac{1}{u(x)} \right) (u(x) - 1)^2 \leq \left( \bar{b} - \frac{1}{1 + (2\bar{b})^{1/2}} \right) (u(x) - 1)^2 \quad \text{for any } x \in [\hat{y}, 1].$$

Thus, when  $\bar{b}$  is small enough such that  $\bar{b} - \frac{1}{1 + (2\bar{b})^{1/2}} < 0$ , we get from the first inequality of (3.5) that

$$0 \leq \int_{\hat{y}}^1 \frac{u+1}{2u} |[(u-1)^2]_x|^2 dx \leq \left( \bar{b} - \frac{1}{1 + (2\bar{b})^{1/2}} \right) \int_{\hat{y}}^1 (u-1)^2 dx < 0, \quad (3.7)$$

which is a contradiction.

In the case  $\tau \ll 1$ , since  $b \leq 1$  and  $1 \leq u \leq u(\hat{y})$ , it follows from (3.4) that

$$\int_{\hat{y}}^1 \frac{u+1}{2u} |[(u-1)^2]_x|^2 dx \leq \int_{\hat{y}}^1 \frac{(u-1)^3}{u} dx - \frac{(u(\hat{y})-1)^3}{3\tau} \leq \left( -\frac{1}{3\tau} \right) (u(\hat{y})-1)^3.$$

Thus, when  $\tau$  is small such that  $\tau < \frac{1}{3}$ , we get a contradiction. Therefore, no interior supersonic solutions exist. The proof is complete.  $\square$

**Theorem 3.3.** *No transonic solution to system (1.5)-(1.6) exists, when the doping profile  $b(x)$  is small such that  $\bar{b}(1 + \sqrt{2\bar{b}}) < 1$ , or the relaxation time  $\tau$  is small such that  $\tau < \frac{1}{3}$ .*

*Proof.* Suppose that  $(\rho, E)$  is a transonic solution separated by a point  $y_0 \in (0, 1)$  in the form

$$\rho(x) = \begin{cases} \rho_{sup}(x), & x \in (0, y_0), \\ \rho_{sub}(x), & x \in (y_0, 1), \end{cases}$$

and

$$\rho_l \rho_r = 1, \quad E_l = E_r \quad \text{with } \rho_l < 1 \text{ and } \rho_r > 1.$$

We first claim

$$E_l = E_r < \frac{1}{\tau}. \quad (3.8)$$

In fact, if  $E_r \geq \frac{1}{\tau}$ , noting the second equation of (1.5) gives

$$(E_{sub})_x(x) = (\rho_{sub} - b) > (1 - b) \geq 0,$$

i.e.  $E_{sub}$  is monotone increasing, we have

$$E_{sub}(x) \geq E_r \geq \frac{1}{\tau}, \quad \text{and} \quad \rho_{sub}(x)E_{sub}(x) - \frac{1}{\tau} > E_r - \frac{1}{\tau} \geq 0, \quad \text{on } (y_0, 1),$$

which in combination with the first equation of (1.5) further gives  $(\rho_{sub})_x(x) > 0$  on  $(y_0, 1)$ .

Thus,  $1 < \rho_r < \rho$  over  $(y_0, 1)$ , which contradicts to  $\rho_{sub}(1) = 1$ . Hence (3.8) holds.

In the case  $b(x) \ll 1$ , multiplying the first equation of (3.2) by  $((u-1)^2)_x$  and integrating the resultant equation over  $(0, y_0)$ , as in (3.4), we get

$$\begin{aligned} & \int_0^{y_0} \frac{u(x)+1}{2u(x)} |[(u(x)-1)^2]_x|^2 dx \\ &= \int_0^{y_0} \left( b(x) - \frac{1}{u(x)} \right) (u(x)-1)^2 dx + (u_l-1)^2 \left( E_l - \frac{2u_l+1}{3\tau} \right) \\ &< \int_0^{y_0} \left( b(x) - \frac{1}{u(x)} \right) (u(x)-1)^2 dx - \frac{2(u_l-1)^3}{3\tau} \\ &< \int_0^{y_0} b(x)(u(x)-1)^2 dx, \end{aligned}$$

where we have used (3.8) in the first inequality. Thus, as in (3.5)-(3.7), when  $\bar{b}$  is small enough such that  $\bar{b} - \frac{1}{1+(2\bar{b})^{1/2}} < 0$ , we get the contradiction

$$\int_0^{y_0} \frac{u+1}{2u} |[(u-1)^2]_x|^2 dx < 0.$$

In the case  $\tau \ll 1$ , since  $\rho_l < 1$ , by (3.8) we get  $\rho_l E_l - 1/\tau < 0$ . Thus,  $\lim_{x \rightarrow y_0^-} (\rho_{sup})_x(x) = (1 - 1/\rho_l^2)^{-1}(\rho_l E_l - 1/\tau) > 0$ . It is then easy to see that  $\rho_{sup}(x)$  attains a local minimal point on  $(0, y_0)$ . Denote by  $\check{y}$  the last local minimal point of  $\rho_{sup}(x)$  on  $(0, y_0)$ , then  $(\rho_{sup})'_x(\check{y}) = 0$ . Set  $u(x) := \frac{1}{\rho_{sup}(x)}$ , then  $u_x(\check{y}) = 0$  and  $u_l = \frac{1}{\rho_l} > 1$ . Hence by the first equation of (3.2), we also get (3.3) at  $\check{y}$ . Multiplying the first equation of (3.2) by  $((u-1)^2)_x$  and integrating the resultant equation over  $(\check{y}, y_0)$ , as shown in (3.4), using (3.3), we get

$$\begin{aligned} & \int_{\check{y}}^{y_0} \frac{u(x)+1}{2u(x)} |[(u(x)-1)^2]_x|^2 dx \\ &= \int_{\check{y}}^{y_0} \left( b(x) - \frac{1}{u(x)} \right) [u(x)-1]^2 dx + (u_l-1)^2 \left( E_l - \frac{2u_l+1}{3\tau} \right) \\ &\quad - (u(\check{y})-1)^2 \left( E(\check{y}) - \frac{2u(\check{y})+1}{3\tau} \right) \\ &= \int_{\check{y}}^{y_0} \left( b(x) - \frac{1}{u(x)} \right) (u(x)-1)^2 dx + (u_l-1)^2 \left( E_l - \frac{1}{\tau} \right) - \frac{2(u_l-1)^3}{3\tau} - \frac{(u(\check{y})-1)^3}{3\tau} \\ &\leq \int_{\check{y}}^{y_0} [u(x)-1]^3 dx - \frac{1}{3\tau} ((u_l-1)^3 + (u(\check{y})-1)^3), \end{aligned}$$

where we have used  $b \leq 1$  and (3.8) in the inequality. Noting

$$\max_{x \in [\check{y}, y_0]} [u(x)-1]^3 = \max\{(u_l-1)^3, (u(\check{y})-1)^3\} =: K,$$

we further have

$$\int_{\check{y}}^{y_0} \frac{u+1}{2u} |[(u-1)^2]_x|^2 dx \leq -\frac{K}{3\tau} < 0 \quad \text{if } \tau < \frac{1}{3}.$$

We thus get a contradiction.  $\square$

### 3.2 Existence of interior supersonic/transonic solutions

In this subsection, we prove the existence of supersonic/transonic solutions when the doping profile is close to the sonic line and the relaxation time is large. The approach adopted is still the compactness technique.

**Theorem 3.4.** *There exists at least one interior supersonic solution to system (1.5)-(1.6) satisfying  $\rho \in C^{\frac{1}{2}}[0, 1]$  and the optimal estimate (1.15), when  $b(x)$  is close to the sonic boundary 1 and the relaxation time is large  $\tau \gg 1$ .*

*Proof.* The proof is similar to that of Theorem 2.3.

*Step 1.* We first consider the Euler-Poisson equations without the semiconductor effect:

$$\begin{cases} \left(1 - \frac{1}{\rho^2}\right) \rho_x = \rho E, \\ E_x = \rho - 1, \\ \rho(0) = \rho(L) = 1 - \delta, \quad (\text{supersonic boundary}), \end{cases} \quad (3.9)$$

where  $L \geq \frac{1}{4}$  is the parameter of length and  $\delta > 0$  is a small constant. Taking  $\underline{b} = 1$  in Lemma 2.4, one can see that (3.9) has a supersonic solution  $(\rho_L, E_L)(x)$  satisfying

$$\beta(L) \leq \underline{\rho} \leq \gamma(L), \quad E_L(0) \geq \sqrt{f(\gamma(L))} > 0, \quad (3.10)$$

where  $\underline{\rho} := \min_{x \in [0, L]} \rho_L(x)$ .

*Step 2.* Let  $\eta$  be a small number to be determined such that  $\delta < \eta \ll 1$ . Denote by  $(\rho_1, E_1)(x)$  the solution of (3.9) with  $L = \frac{1}{2}$ . Now let us consider the ODE system with the semiconductor effect  $-\frac{1}{\tau}$  and a small perturbation of the doping profile around the sonic line, i.e.  $b(x) = 1 - \epsilon e(x)$ :

$$\begin{cases} \left(1 - \frac{1}{\rho^2}\right) \rho_x = \rho E - \frac{1}{\tau}, \\ E_x = \rho - 1 + \epsilon e(x), \\ (\rho(0), E(0)) = (1 - \delta, E_1(0)). \end{cases} \quad (3.11)$$

Here  $\tau \gg 1$ ,  $0 < \epsilon \ll 1$ ,  $0 \leq e(x) \in L^\infty(\mathbb{R}^+)$ , and we have extended periodically the doping profile  $b$  to  $\mathbb{R}^+$ . We claim that there exists a number  $y_1 \leq C\eta$  such that  $\rho(y_1) = 1 - \eta$ , where  $C > 0$  is a constant independent of  $\tau$ ,  $\epsilon$ ,  $\delta$  and  $\eta$ .

It is easy to see that if  $\tau \geq \frac{4}{E_1(0)}$  and  $\delta \leq \frac{1}{4}$ , then the initial data of (3.11) satisfies

$$\rho(0)E(0) - \frac{1}{\tau} = (1 - \delta)E_1(0) - \frac{1}{\tau} \geq \frac{E_1(0)}{2} > 0.$$

From the first equation of (3.11), we know that  $\rho$  is decreasing in a neighborhood of 0. If  $\rho$  keeps decreasing on  $[0, x]$ , then

$$E(x) = E_1(0) + \int_0^x (\rho - 1 + \epsilon e(s)) ds \geq E_1(0) - x, \quad (3.12)$$



which indicates that if

$$x \leq \frac{E_1(0)}{4}, \quad (3.13)$$

then

$$E(x) \geq E_1(0) - \frac{E_1(0)}{4} = \frac{3E_1(0)}{4}.$$

We next prove that if  $\rho$  keeps decreasing, denoting by  $y_1$  the first number that  $\rho$  attains  $1 - \eta$ , then  $y_1 \leq C\eta^2$  for some constant  $C > 0$ . In fact, observing that

$$\rho_x = \frac{\rho E - \frac{1}{\tau}}{1 - \frac{1}{\rho^2}} = \frac{\rho^2(\rho E - \frac{1}{\tau})}{\rho^2 - 1} \leq \frac{\rho^3 E}{\rho^2 - 1} \leq -\frac{3(1 - \eta)^3 E_1(0)}{4\eta(2 - \eta)} \leq -\frac{E_1(0)}{16\eta}, \text{ if } \eta \leq \frac{1}{2}.$$

Thus,

$$y_1 = \frac{\delta - \eta}{\int_0^1 \rho_x(sy_1)ds} \leq \frac{16\eta^2}{E_1(0)}.$$

Hence, if  $\eta \leq \frac{E_1(0)}{8}$ , then (3.13) holds, and  $\rho$  keeps decreasing and attains  $1 - \eta$  at  $y_1$  with  $y_1 \leq \frac{16\eta^2}{E_1(0)}$ . By (3.12),

$$E_1(0) - C\eta^2 \leq E(y_1) \leq E_1(0) + C\eta^2. \quad (3.14)$$

*Step 3.* Now let us reconsider the ODE system without the semiconductor effect

$$\begin{cases} \left(1 - \frac{1}{\hat{\rho}^2}\right) \hat{\rho}_x = \hat{\rho} \hat{E}, \\ \hat{E}_x = \hat{\rho} - 1, \\ (\hat{\rho}(0), \hat{E}(0)) = (1 - \delta, \hat{E}_0). \end{cases} \quad (3.15)$$

Taking  $b = 1$  in step 2 in the proof of Theorem 2.3, we know that there exist  $\hat{E}_0 \in (\frac{E_1(0)}{2}, 2E_1(0))$  and  $y_2 \leq C\eta^2$  such that (3.15) has a supersonic solution  $(\hat{\rho}, \hat{E})$  satisfying

$$\hat{\rho}(y_2) = 1 - \eta, \quad \hat{E}(y_2) = E(y_1), \quad E_1(0) - C\eta^2 \leq \hat{E}(y_2) \leq E_1(0) + C\eta^2. \quad (3.16)$$

Here  $E$  and  $y_1$  are given by step 2. Moreover, the length  $\hat{L}$  of the solution of (3.15) with initial boundary data  $(\hat{\rho}(0), \hat{E}(0)) = (1 - \delta, \hat{E}_0)$ ,  $\hat{\rho}(\hat{L}) = 1 - \delta$  satisfies

$$\frac{1}{4} \leq \hat{L} \leq \frac{3}{4}.$$

*Step 4.* Set  $(\bar{\rho}, \bar{E})(x) := (\hat{\rho}, \hat{E})(x - y_1 + y_2)$ , then  $(\bar{\rho}, \bar{E})$  satisfies (3.9) with initial-boundary data

$$(\bar{\rho}, \bar{E})(y_1) = (1 - \eta, \hat{E}(y_2)) = (\rho, E)(y_1) \quad \text{and} \quad \bar{\rho}(y_3) = 1 - \eta$$

with  $y_3 := \hat{L} + y_1 - 2y_2$ . As in step 3 of the proof of Theorem 2.3, when  $\tau \gg 1$  and  $0 < \epsilon \ll 1$  such that  $C(\frac{1}{\tau^2} + \epsilon^2)e^{C/\eta^2} \leq 1/4$ , system (3.11) has a unique solution  $(\rho, E)$  on  $[0, y_3]$  satisfying

$$\rho(y_3) \leq 1 - \frac{\eta}{2}, \quad E(y_3) \leq E_1(0) + C\eta. \quad (3.17)$$

Now taking  $y_3$  as the initial data, as in step 3 of the proof of Theorem 2.3, we can extend  $(\rho, E)$ , the solution of (3.11), to the state  $\rho = 1 - \delta$ . Denote by  $y_4$  the number that  $\rho(y_4) = 1 - \delta$ , then

$$\rho(0) = \rho(y_4) = 1 - \delta, \quad E(0) = E_1(0), \quad E(y_4) \leq E_1(0) + C\eta. \quad (3.18)$$

Moreover,

$$\frac{1}{4} - C\eta^2 \leq y_4 \leq \frac{3}{4} + C\eta^2.$$

Now we take  $L = \frac{3}{2}$  in (3.9) and denote by  $(\rho_2, E_2)$  its solution. Applying a similar argument above, we know that there exists an interval  $[0, y_5]$  with

$$\frac{5}{4} - C\eta^2 \leq y_5 \leq \frac{7}{4} + C\eta^2,$$

such that system (3.11) has a solution on  $[0, y_5]$  satisfying

$$\rho(0) = \rho(y_5) = 1 - \delta, \quad E(0) = E_2(0), \quad E(y_5) \leq -E_2(0) + C\eta. \quad (3.19)$$

Without loss of generality, we assume that  $E_1(0) < E_2(0)$ , then when  $\eta \ll 1$ , for any initial data  $E_L(0) \in (E_1(0), E_2(0))$ , (3.11) has a solution. Noting the length parameter  $L$  is continuous with respect to the initial data, system (3.11) has a solution on  $[0, 1]$  satisfying  $\rho(0) = \rho(1) = 1 - \delta$  and  $E(0) \in (E_1(0), E_2(0))$ .

*Step 5.* For any  $\delta > 0$ , denote by  $(\rho^\delta, E^\delta)$  the solution of (3.11) with boundary data  $\rho^\delta(0) = \rho^\delta(1) = 1 - \delta$ . The velocity  $u^\delta = 1/\rho^\delta$  satisfies

$$\left( \left( u^\delta - \frac{1}{u^\delta} \right) (u^\delta)_x \right)_x + \frac{(u^\delta)_x}{\tau} - \left( \frac{1}{u^\delta} - b \right) = 0, \quad u^\delta(0) = u^\delta(1) = \frac{1}{1 - \delta}. \quad (3.20)$$

Multiplying (3.20) by  $\left( u^\delta - \frac{1}{1 - \delta} \right)^2$ , as in (3.13), we have

$$\begin{aligned} & \frac{2\delta}{1 - \delta} \int_0^1 \frac{(u^\delta + 1)}{u^\delta} \left( u^\delta - \frac{1}{1 - \delta} \right) |u_x^\delta|^2 dx + \int_0^1 \frac{(u^\delta + 1)}{2u^\delta} \left| \left( \left( u^\delta - \frac{1}{1 - \delta} \right)^2 \right)_x \right|^2 \\ &= \int_0^1 \left( b - \frac{1}{u^\delta} \right) \left( u^\delta - \frac{1}{1 - \delta} \right)^2 \\ &\leq \frac{1}{2} \int_0^1 \left( u^\delta - \frac{1}{1 - \delta} \right)^4 + \frac{1}{2} \int_0^1 \left( b - \frac{1}{u^\delta} \right)^2, \\ &\leq \frac{1}{4} \int_0^1 \left| \left( \left( u^\delta - \frac{1}{1 - \delta} \right)^2 \right)_x \right|^2 + \frac{1}{2} \int_0^1 b^2, \end{aligned}$$

which gives

$$\left\| \left( u^\delta - \frac{1}{1 - \delta} \right)^2 \right\|_{H^1} \leq C,$$

and hence

$$\|u^\delta\|_{L^\infty} \leq C.$$

It then follows that

$$\rho^\delta = \frac{1}{u^\delta} \geq \frac{1}{\|u^\delta\|_{L^\infty}} \geq \frac{1}{C}, \text{ and } \|(1 - \delta - \rho^\delta)^2\|_{H^1} \leq C.$$

Therefore, there exists a function  $\rho^0$  such that, as  $\delta \rightarrow 1^+$ , up to a subsequence,

$$\begin{aligned} (1 - \rho^\delta)^2 &\rightharpoonup (1 - \rho^0)^2 \text{ weakly in } H^1(0, 1), \\ (1 - \rho^\delta)^2 &\rightarrow (1 - \rho^0)^2 \text{ strongly in } C^{\frac{1}{2}}[0, 1]. \end{aligned} \tag{3.21}$$

Applying the same procedure as the proof of Theorem 2.1, one can show that  $\rho^0$  is the supersonic solution of (1.7).  $\square$

**Theorem 3.5.** *There exist infinitely many transonic solutions to (1.5)-(1.6), when  $b(x)$  is close to the sonic boundary 1 and  $\tau \gg 1$ .*

*Proof.* The proof is similar to that of Theorem 2.3.

*Step 1.* Consider the ODE system (3.11), in view of step 4 in the proof of Theorem 3.4, given small constants  $\eta \ll 1$  ( $\delta < \eta$ ),  $\epsilon \ll 1$ ,  $\tau \gg 1$ , (3.11) has a supersonic solution  $(\rho, E)$  on  $[0, y_4]$  satisfying

$$\frac{1}{4} - C\eta^2 \leq y_4 \leq \frac{3}{4} + C\eta^2, \quad \rho(0) = \rho(y_4) = 1 - \delta, \quad E(0) = E_1(0), \quad E(y_4) \leq -E_1(0) + C\eta,$$

where  $E_1$  is the solution of (3.9) with  $L = \frac{1}{2}$ . Setting  $\rho_l = 1 - \eta$  and taking the jump location  $\bar{y}_0 \in (0, y_4)$  by the last number when  $\rho(\bar{y}_0) = \rho_l$ , we focus this supersonic solution  $(\rho_{sup}, E_{sup})(x)$  only on  $[0, \bar{y}_0]$ . As in step 4 of the proof of Theorem 2.3, when  $\eta \ll 1$  such that  $(C + E_1(0))\eta \leq \frac{E_1(0)}{2}$  and  $C\eta^2 < \frac{E_1(0)}{4}$ , then

$$\rho_r E_r - \frac{1}{\tau} \leq -\frac{E_1(0)}{4} < 0.$$

From the first equation of (3.11), we know such an initial value problem has a decreasing subsonic solution in a neighborhood of  $\bar{y}_0^+$ . We denote this subsonic solution by  $(\rho_{sub}, E_{sub})(x)$ . If  $\rho_{sub}$  keeps decreasing, then

$$\begin{aligned} E_{sub}(x) &= E_r + \int_{\bar{y}_0}^x (\rho_{sub} - 1 + \epsilon e(x)) dx \\ &\leq -E_1(0) + C\eta + \int_{\bar{y}_0}^x (\rho_r - 1 + \epsilon e(x)) dx \\ &\leq -E_1(0) + C\eta + (x - \bar{y}_0) \left( \frac{\eta}{1 - \eta} + \epsilon \|e\|_{L^\infty} \right), \end{aligned}$$

which implies that if  $C\eta \leq \min(\frac{E_1(0)}{2}, \frac{1}{2})$ ,  $\epsilon \leq 1$  and

$$x - \bar{y}_0 \leq \frac{4}{(1 + \|e\|_{L^\infty})E_1(0)}, \tag{3.22}$$

then

$$E_{sub}(x) < -\frac{E_1(0)}{4} < 0.$$

We now claim that if  $\rho_{sub}$  keeps decreasing, denoting by  $y_6$  the number that  $\rho_{sub}$  attains  $1 + \delta$ , then  $y_6 - \bar{y}_0 \leq C\eta$ .

In fact, observing that

$$(\rho_{sub})_x = \frac{\rho_{sub}E_{sub} - \frac{1}{\tau}}{1 - \frac{1}{\rho_{sub}^2}} \leq -\frac{(1-\eta)^2 E_1(0)}{4\eta(2-\eta)} < -\frac{(1-\eta)^2 E_1(0)}{4\eta},$$

we get

$$y_6 - \bar{y}_0 = \frac{\delta - \frac{\eta}{1-\eta}}{\int_0^1 (\rho_{sub})_x (sy_6 + (1-s)\bar{y}_0) ds} \leq \frac{\eta}{1-\eta} \cdot \frac{4\eta}{E_1(0)(1-\eta)^2} \leq 32\eta \text{ if } \eta < \min(E_1(0), \frac{1}{2}).$$

Obviously, if  $\eta < \frac{1}{16(1+\|e\|_{L^\infty})E_1(0)}$ , then (3.22) holds and  $\rho_{sub}$  keeps decreasing and attains  $1 + \delta$  at  $y_6$ . Now we have constructed the transonic solution to (1.5) in  $[0, y_6]$  with  $\frac{1}{4} - C\eta \leq y_6 \leq \frac{3}{4} + C\eta$  as follows

$$(\rho_{trans}, E_{trans})(x) = \begin{cases} (\rho_{sup}, E_{sup})(x), & x \in [0, \bar{y}_0), \\ (\rho_{sub}, E_{sub})(x), & x \in (\bar{y}_0, y_6], \end{cases}$$

which satisfies the boundary condition

$$\rho_{sup}(0) = 1 - \delta, \quad \rho_{sub}(y_6) = 1 + \delta,$$

and the entropy condition at  $\bar{y}_0$

$$0 < \rho_{sup}(\bar{y}_0^-) = 1 - \eta < 1 < \rho_{sub}(\bar{y}_0^+),$$

and the Rankine-Hugoniot condition (1.11) at  $\bar{y}_0$ .

*Step 2.* Denote by  $(\rho_2, E_2)$  the solution of (3.9) with  $L = \frac{3}{2}$ , by step 4 of the proof of Theorem 3.4, (3.11) has a supersonic solution  $(\rho, E)$  on  $[0, y_7]$  with

$$\frac{5}{4} - C\eta^2 \leq y_7 \leq \frac{7}{4} + C\eta^2, \quad \rho(0) = \rho(y_7) = 1 - \delta, \quad E(0) = E_2(0), \quad E(x_{10}) \leq -E_2(0) + C\eta.$$

As in step 1, we may construct another transonic solution for (1.5) in the form of

$$(\rho_{trans}, E_{trans})(x) = \begin{cases} (\rho_{sup}, E_{sup})(x), & x \in [0, \tilde{y}_0), \\ (\rho_{sub}, E_{sub})(x), & x \in (\tilde{y}_0, y_7], \end{cases}$$

where  $\tilde{y}_0 \in (0, x_{10})$  and  $\frac{5}{4} - C\eta^2 \leq y_7 \leq \frac{7}{4} + C\eta^2$  are some determined numbers. This transonic solution satisfies the boundary condition

$$\rho_{sup}(0) = 1 - \delta, \quad \rho_{sub}(y_7) = 1 + \delta,$$

the entropy condition at  $\tilde{y}_0$

$$0 < \rho_{sup}(\tilde{y}_0^-) = 1 - \eta < 1 < \rho_{sub}(\tilde{y}_0^+),$$

and the Rankine-Hugoniot condition (1.11) at  $\tilde{y}_0$ .

*Step 3.* Without loss of generality, we assume that  $E_1(0) < E_2(0)$ . As in step 6 in the proof of Theorem 2.3, for any  $E_0 \in (E_1(0), E_2(0))$ , (3.11) has a transonic solution on an interval  $[0, y_8]$ . Applying the continuation argument in the length of the interval, one can see that for any  $\delta > 0$  (1.5)-(1.6) has a transonic solution denote by  $(\rho_{trans}^\delta, E_{trans}^\delta)$  on  $[0, 1]$ , and it satisfies the boundary conditions

$$\rho_{sup}^\delta(0) = 1 - \delta, \quad \rho_{sub}^\delta(1) = 1 + \delta,$$

the entropy condition

$$0 < \rho_{sup}^\delta(y_0^\delta) = 1 - \eta < 1 < \rho_{sub}^\delta(y_0^\delta),$$

and the Rankine-Hugoniot condition (1.11) at a jump location  $y_0^\delta$  in  $(0, 1)$ . Letting  $\delta \rightarrow 0^+$ , applying the diagonal argument for  $(\rho_{trans}^\delta, E_{trans}^\delta)$ , we know that (1.5)-(1.6) has a transonic solution  $(\rho_{trans}, E_{trans})(x)$  for  $x \in [0, 1]$  and satisfies the sonic boundary condition, the entropy condition and the Rankine-Hugoniot condition at a jump location  $y_0$  in  $(0, 1)$ .

Because  $\tau$  and  $\epsilon$  only depend on  $(E_1(0), E_2(0), \eta)$ , and  $\eta$  only depends on  $(E_1(0), E_2(0))$ , there exists a  $\eta_0 > 0$  such that for any  $\eta \in (0, \eta_0)$ , there exists a transonic solution jumps at  $\rho_l = 1 - \eta$ . Thus, we obtain infinitely many transonic solutions due to arbitrary choice of  $0 < \eta < \eta_0$ .  $\square$

*Proof of Theorem 1.2.* Combining Theorems 3.1-3.5, we immediately obtain Theorem 1.2.  $\square$

## 4 Concluding remarks

In this section, we remark that Theorems 1.1-1.2 both hold for the isentropic hydrodynamic model. For isentropic flow, the pressure function satisfies  $P(\rho) = T\rho^\gamma$  for some constants  $T > 0$  and  $\gamma > 1$ . Then system (1.3) reduces to

$$\begin{cases} J = \text{constant}, \\ \left( \frac{J^2}{\rho} + T\rho^\gamma \right)_x = \rho E - \frac{J}{\tau}, \\ E_x = \rho - b(x). \end{cases} \quad x \in (0, 1). \quad (4.1)$$

The sonic flow means

$$\text{fluid velocity: } u = \frac{J}{\rho} = c = \sqrt{P'(\rho)} = \sqrt{T\gamma\rho^{\gamma-1}} : \text{ sound speed.}$$

Without loss of generality, we assume that

$$J = T\gamma = 1.$$

Then (4.1) is transformed to

$$\begin{cases} \left( \rho^{\gamma-1} - \frac{1}{\rho^2} \right) \rho_x = \rho E - \frac{1}{\tau}, \\ E_x = \rho - b(x), \end{cases} \quad (4.2)$$

and our sonic boundary conditions are proposed as

$$\rho(0) = \rho(1) = 1. \quad (4.3)$$

Now as in the isothermal fluid, we can also identify that, for system (4.2),  $\rho > 1$  is for the subsonic flow,  $\rho = 1$  stands for the sonic flow, and  $0 < \rho < 1$  represents for the supersonic flow.

Following the proofs of Theorems 1.1-1.2, one can easily obtain the following classification of solutions to system (4.2)-(4.3) for isentropic flow:

**Theorem 4.1.**

1. For subsonic doping profile:  $b(x) \in L^\infty(0, 1)$  and  $b(x) > 1$  in  $[0, 1]$ , then system (4.2)-(4.3) admit:
  - (a) a unique pair of interior subsonic solutions  $(\rho_{sub}, E_{sub})(x) \in C^{\frac{1}{2}}[0, 1] \times H^1(0, 1)$  with  $\rho_{sub}(x) \geq 1$ ;
  - (b) at least a pair of interior supersonic solutions  $(\rho_{sup}, E_{sup})(x) \in C^{\frac{1}{2}}[0, 1] \times H^1(0, 1)$  with  $\rho_{sup}(x) \leq 1$ ;
  - (c) if further that  $\tau$  is large and that  $\bar{b} - \underline{b} \ll 1$ , then system (4.2)-(4.3) has infinitely many transonic shock solutions  $(\rho_{trans}, E_{trans})(x) \in L^\infty(0, 1) \times C^0(0, 1)$ ;
  - (d) if further that  $b(x) = b > 1$  is a constant, then when  $\tau$  is small enough, (4.2)-(4.3) has infinitely many  $C^1$  transonic solution; moreover, in this case there is no transonic shock solution.
2. For supersonic doping profile:  $b(x) \in L^\infty(0, 1)$  and  $0 < b(x) \leq 1$  in  $[0, 1]$ , then (4.2)-(4.3) admit:
  - (a) a pair of interior supersonic solutions  $(\rho_{sup}, E_{sup})(x) \in C^{\frac{1}{2}}[0, 1] \times H^1(0, 1)$  and infinitely many transonic shock solution  $(\rho_{trans}, E_{trans})(x) \in L^\infty(0, 1) \times C^0(0, 1)$  if  $b(x)$  is close to 1 and  $\tau$  is large enough;
  - (b) no interior subsonic solutions  $(\rho_{sub}, E_{sub})(x)$ ;
  - (c) no interior supersonic solutions  $(\rho_{sup}, E_{sup})(x)$ , nor transonic shock solutions  $(\rho_{trans}, E_{trans})(x)$  if  $b(x)$  is small or  $\tau$  is small.

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